

## C&O 739, Winter 2009 Assignment 2

*Due Tuesday, March 3, in class.*

1. Let  $E$  be a vector space of dimension  $n$ , and let  $E' \subset E$  be a subspace of dimension  $m$ . Since every  $d$ -dimensional subspace of  $E'$  is also a  $d$ -dimensional subspace of  $E$ , there is a natural inclusion

$$i : \text{Gr}_d(E') \hookrightarrow \text{Gr}_d(E).$$

Write  $\sigma_\lambda$  for a Schubert class in  $H^*(\text{Gr}_d(E))$  and  $\sigma'_\lambda$  for a Schubert class in  $H^*(\text{Gr}_d(E'))$ .

- (a) If  $F'_\bullet$  is a flag in  $E'$ , show that  $i(\Omega_\lambda(F'_\bullet)) = \Omega_{\lambda_+}(F_\bullet)$ , for some partition  $\lambda_+$  and some flag  $F_\bullet$  in  $E$ . Determine the relationship between  $\lambda$  and  $\lambda_+$ .
- (b) From part (a), it follows that the map  $i_* : H_*(\text{Gr}_d(E')) \rightarrow H_*(\text{Gr}_d(E))$  is given by  $i_*(\sigma'_\lambda) = \sigma_{\lambda_+}$ . (Here we are using the Poincaré isomorphism to identify homology and cohomology classes. Note that  $i_*$  is *not* a ring homomorphism.)

Compute  $i_*(1) = i_*([\text{Gr}_d(E')])$ , and show that  $\sigma_{\lambda_+} = i_*(1) \cdot \sigma_\lambda$ .

- (c) It is a general fact from topology that if  $f : X \rightarrow Y$  is a smooth map of closed, oriented manifolds, then for  $\alpha \in H^*(X)$ ,  $\beta \in H^*(Y)$ ,

$$\int_Y f_*(\alpha) \cdot \beta = \int_X \alpha \cdot f^*(\beta).$$

Prove that

$$i^*(\sigma_\lambda) = \begin{cases} \sigma'_\lambda & \text{if } \lambda \in \text{Part}_{d,m} \\ 0 & \text{otherwise.} \end{cases}$$

2. Let  $L_1, L_2, L_3$  be the row spaces of the following matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Let  $M_{\alpha,\beta}$  be row space of the matrix

$$\begin{pmatrix} 1 & 0 & 1 & \alpha \\ 0 & 1 & \beta & 1 \end{pmatrix}$$

where  $(\alpha, \beta) \in \mathbb{C}^2$ .

- (a) Determine conditions on  $(\alpha, \beta)$  so that the four special Schubert varieties in  $\text{Gr}_2(\mathbb{C}^4)$ ,

$$\Omega_{(1)}(L_1), \Omega_{(1)}(L_2), \Omega_{(1)}(L_3), \Omega_{(1)}(M_{\alpha,\beta})$$

are transverse. For these  $(\alpha, \beta)$ , find explicitly all  $V \in \text{Gr}_2(\mathbb{C}^4)$  such that

$$V \in \Omega_{(1)}(L_1) \cap \Omega_{(1)}(L_2) \cap \Omega_{(1)}(L_3) \cap \Omega_{(1)}(M_{\alpha,\beta}).$$

(b) Show that there is a loop

$$t \in [0, 1] \mapsto (\alpha_t, \beta_t) \in \mathbb{C}^4$$

(i.e. a path such that  $(\alpha_0, \beta_0) = (\alpha_1, \beta_1)$ ) with the following properties:

- $\Omega_{(1)}(L_1), \Omega_{(1)}(L_2), \Omega_{(1)}(L_3), \Omega_{(1)}(M_{\alpha_t, \beta_t})$  are transverse for all  $t \in [0, 1]$ .
- For any path

$$t \in [0, 1] \mapsto V_t \in \Omega_{(1)}(L_1) \cap \Omega_{(1)}(L_2) \cap \Omega_{(1)}(L_3) \cap \Omega_{(1)}(M_{\alpha_t, \beta_t}),$$

we have  $V_0 \neq V_1$ , (i.e. the monodromy of the loop permutes the points of intersection).

3. The iterated Littlewood-Richardson rule states:

Let  $\lambda_1, \dots, \lambda_s \in \text{Part}_{d,n}$  be such that  $|\lambda_1| + \dots + |\lambda_s| = d(n-d)$ . Then

$$\int_{\text{Gr}_d(E)} \sigma_{\lambda_1} \cdots \sigma_{\lambda_s}$$

is the number of pairs of sequences  $(\mu_0, \mu_1, \dots, \mu_s), (T_1, \dots, T_s)$ , where  $\mu_i \in \text{Part}_{d,n}$  are partitions such that

$$0 = \mu_0 \leq \mu_1 \leq \dots \leq \mu_s = (n-d)^d$$

and each  $T_i$  is a Littlewood-Richardson tableau of shape  $\mu_i / \mu_{i-1}$  and weight  $\lambda_i$ .

- Use the Littlewood-Richardson rule as stated in class to prove the iterated Littlewood-Richardson rule.
- Use the iterated Littlewood-Richardson rule to prove that

$$h_{\nu_1} \cdots h_{\nu_d} = \sum_{\mu} K_{\nu}^{\mu} s_{\mu}$$

where  $K_{\nu}^{\mu}$  are the Kostka numbers. (Here  $\nu_1 \geq \dots \geq \nu_d$  are the parts of  $\nu$ . The Kostka number  $K_{\nu}^{\mu}$  is defined to be the number of semistandard Young tableaux of shape  $\mu$  and weight  $\nu$ .)