# C&O 739, Winter 2009
# Assignment 1

*Due Tuesday, February 10, in class.*

1. Recall the Plücker relations are:

$$p_{i_1,\ldots,i_d}\, p_{j_1,\ldots,j_{d'}} - \sum p_{i_1,\ldots,j_1,\ldots,j_\ell,\ldots i_d}\, p_{i_{m_1},\ldots,i_{m_\ell},j_{\ell+1},\ldots,j_{d'}} = 0$$

   where $d \geq d'$ and $\ell \geq 1$.

   (a) Show that for $\ell = 1$, these equations can be rewritten as:

$$\sum_{m=1}^{d+1} (-1)^m p_{i_1,\ldots,\widehat{i_m},\ldots,i_{d+1}}\, p_{j_1,\ldots,j_{d'-1},i_m} = 0$$

   (b) Show that all Plücker relations can be infered from the $\ell = 1$ relations.

2. Prove the following:

   (a) $\Omega_\lambda = \coprod_{\mu \geq \lambda} \Omega^\circ_\mu$.

   (b) $\Omega_\lambda \setminus \Omega^\circ_\lambda = \bigcup_{\mu \succ \lambda} \Omega_\mu$.

   (c) $\Omega_\lambda$ is the closure of $\Omega^\circ_\lambda$.

3. Let $S_d$ be the homogenous coordinate ring of the Grassmannian $\mathrm{Gr}_d(E)$, generated by Plücker variables $p_\lambda$, $\lambda \in \mathrm{Part}_{d,n}$. Let $I_\lambda \subset S_d$ be the ideal

$$I_\lambda = \langle p_\mu \mid \mu \not\geq \lambda \rangle\,.$$

   Show that $\Omega_\lambda = \mathrm{Proj}(S_d/I_\lambda)$.

   (In other words, check that the points on $\Omega_\lambda$ are exactly those subspaces $V$ whose Plücker coordinates $p_\mu(V)$ are zero for all $\mu \not\leq \lambda$. You do not need to show that $S_d/I_\lambda$ is an integral domain; this is true but considerably harder.)

4. Let $F_\bullet$ be a full flag in $E$.

   (a) Show that the group $B := \mathrm{Stab}_{GL(E)}(F_\bullet)$ is isomorphic the group of invertible upper (or lower) triangular $n \times n$ matrices.

   (b) Show that every Schubert cell $\Omega^\circ_\lambda(F_\bullet) \subset \mathrm{Gr}_d(E)$ is an orbit of the group $B$ acting on $\mathrm{Gr}_d(E)$.