Math 245, Spring 2012
Assignment 9

Not to be handed in.

1. Let $A \in \mathsf{M}_{n\times n}(\mathbb{R})$ be a skew-symmetric matrix. Show that e^A is an orthogonal matrix. (You may use the fact that $e^{X+Y} = e^X e^Y$ if $XY = YX$.)

2. Let $Q \in \mathsf{M}_{n\times n}(\mathbb{R})$ be an orthogonal matrix.

   (a) Prove that $\det(Q) = \pm 1$.

   (b) If $\det(Q) = -1$, prove that -1 is an eigenvalue of Q.

   (c) If n is odd and $\det(Q) = 1$, prove that 1 is an eigenvalue of Q.

3. Let $\mathbb{H}$ be the set of all 2×2 matrices of the form

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \qquad a, b \in \mathbb{C}\,.$$

   $\mathbb{H}$ is a 4-dimensional vector space over the real numbers; elements of $\mathbb{H}$ are called **matrix quaternions**.

   (a) Show that the product of two matrix quaternions is a matrix quaternion. Show that a non-zero matrix quaternion is invertible, and its inverse is a matrix quaternion.

   (b) Define $\langle q_1, q_2\rangle = \frac{1}{2}\mathrm{Re}(\mathrm{tr}\, q_1 q_2^*)$, for $q_1, q_2 \in \mathbb{H}$. Show that $\langle\cdot,\cdot\rangle$ is an inner product on $\mathbb{H}$.

   (c) For $q \in \mathbb{H}$, let $L_q, R_q \in \mathsf{L}(\mathbb{H})$ be the operators $L_q(x) = qx$, and $R_q(x) = xq$. Show that L_q and R_q are normal operators.

   (d) Let $W = \mathrm{span}\{I_2, q\}$, where $q \in \mathbb{H}$ is a non-zero vector. Let θ be the angle between q and I_2, and let $T_q = L_q R_q^{-1}$. Show that T_q is a rotation that fixes every vector in W and rotates $W^\perp$ through an angle of 2θ.

   (e) Show that $T_q = T_{q'}$ if and only if $q = cq'$ for some $c \in \mathbb{R} \setminus \{0\}$.

   (f) Let V denote the 3-dimensional real vector space of 2×2 skew-hermitian matrices. It is easy to check that $V = \{I_2\}^\perp \subset \mathbb{H}$. Show that V is an invariant subspace for each operator T_q, and that the induced operator $(T_q)_V$ is a rotation of V. Show that every rotation of V is of the form $(T_q)_V$ for some $q \in \mathbb{H}$.

   (*Note:* The set of rotations of $\mathbb{R}^3$ is commonly denoted $\mathrm{SO}(3,\mathbb{R})$ (SO stands for "special orthogonal group"). The set of 1-dimensional subspaces of $\mathbb{R}^4$ is commonly denoted $\mathbb{RP}^3$ (called "real projective 3-space"). This argument shows that there is a natural identification between $\mathrm{SO}(3,\mathbb{R})$ and $\mathbb{RP}^3$. This is generally considered to be a bit of a remarkable coincidence: it is a special property of 3-dimensional rotations that doesn't generalize to higher dimensions. Nevertheless, it is a well-known, useful fact, and although it doesn't generalize, there are a number of other coincidences of a similar flavour. For example, rotations of $\mathbb{R}^4$ can also be understood in terms of matrix quaternions: every rotation of $\mathbb{H}$ is of the form $L_q R_{q'}$ where q, q' are unitary.)