Math 245, Spring 2012
Assignment 8

Due Friday July 6, in class.

1. Let $A \in \mathsf{M}_{n\times n}(\mathbb{F})$ be a self-adjoint matrix (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$). Prove that A is positive-definite if and only if all eigenvalues of A are positive real numbers.

2. Let V be an n-dimensional inner product space over $\mathbb{C}$, with inner product $\langle \cdot, \cdot \rangle$. Define $V_{\mathbb{R}}$ to be the $2n$-dimensional vector space over $\mathbb{R}$, obtained by restricting restricting scalar multiplication in V to the field of real numbers. For example, if $V = \mathbb{C}^2$, then $V_{\mathbb{R}} = \mathbb{C}^2$ but a basis for $V_{\mathbb{R}}$ is $\{(1,0),(0,1),(i,0),(0,i)\}$. Any linear operator $T \in \mathsf{L}(V)$ also defines an linear operator $T_{\mathbb{R}} \in \mathsf{L}(V_{\mathbb{R}})$.

   (a) Prove that $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ defined by $\langle x, y\rangle_{\mathbb{R}} = \mathrm{Re}\langle x, y\rangle$ is an inner product on $V_{\mathbb{R}}$.

   (b) If $\beta = \{x_1, \ldots, x_n\}$ is an orthonormal basis for V, prove that $\beta_{\mathbb{R}} = \{x_1, \ldots, x_n, ix_1, \ldots, ix_n\}$ is an orthonormal basis for $V_{\mathbb{R}}$.

   (c) Prove the following: T is normal if and only if $T_{\mathbb{R}}$ is normal; T is self-adjoint if and only if $T_{\mathbb{R}}$ is self-adjoint; T is unitary if and only if $T_{\mathbb{R}}$ is orthogonal.

3. Let A, B be $n \times n$ Hermitian matrices, and let $C = A + B$. Since the eigenvalues of A, B and C are real, we can list them in decreasing order.

   – Let $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ be the eigenvalues of A.
   – Let $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ be the eigenvalues of B.
   – Let $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ be the eigenvalues of C.

   (If an eigenvalue has multiplicity m, it appears m times in the list.)

   (a) Prove that for any $x \in \mathbb{C}^n$, $\langle Ax, x\rangle \leq \alpha_1 \|x\|^2$, and determine when equality occurs.

   (b) Prove that $\gamma_1 \leq \alpha_1 + \beta_1$. Determine when equality occurs.
   (*Hint:* Use part (a), where x is an eigenvector for C with eigenvalue γ_1.)

   (c) Give an example to show that part (b) may be false if A, B and C have real eigenvalues, but are not necessarily Hermitian.

   (d) Prove that $\gamma_k \leq \alpha_i + \beta_{k-i+1}$, for all $1 \leq i \leq k \leq n$.
   (*Hint:* Let $W = \mathsf{E}_{\alpha_i} + \mathsf{E}_{\alpha_{i+1}} + \cdots + \mathsf{E}_{\alpha_n}$. What can we say about $\langle Ax, x\rangle$ if $x \in W$?)