Math 245, Spring 2012
Assignment 6

Due Friday, June 22, in class.

Throughout this assignment, the field $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

1. Let V be a vector space over $\mathbb{F}$. Let $\langle \cdot, \cdot \rangle$, and $\langle \cdot, \cdot \rangle'$ be two inner products on V. For $\alpha \in \mathbb{R}$, define
$$\langle x, y \rangle_\alpha = \langle x, y \rangle + \alpha \langle x, y \rangle' .$$
   (a) If $\alpha > 0$, prove that $\langle \cdot, \cdot \rangle_\alpha$ is an inner product on V.
   (b) Show that if $\dim V \neq 0$, there exists an $\alpha < 0$ such that $\langle \cdot, \cdot \rangle_\alpha$ is not an inner product on V.
   (c) Determine whether the following statement is always true: There exists an $\alpha < 0$ such that $\langle \cdot, \cdot \rangle_\alpha$ is an inner product on V. Prove it, or give a counterexample.

2. Let $\beta = \{v_1, \ldots, v_n\}$ be an orthonormal basis for an inner product space V over $\mathbb{F}$.
   (a) Prove that for all $x \in V$,
$$[x]^\beta = \begin{pmatrix} \langle x, v_1 \rangle \\ \langle x, v_2 \rangle \\ \vdots \\ \langle x, v_n \rangle \end{pmatrix} .$$
   (b) Prove that for all $x, y \in V$,
$$\langle x, y \rangle = \left\langle [x]^\beta, [y]^\beta \right\rangle_{\mathbb{F}^n}$$
   where the inner product on the right hand side is the standard inner product on $\mathbb{F}^n$.

3. Let V be an inner product space over $\mathbb{F}$.
   (a) If $\mathbb{F} = \mathbb{R}$, prove that
$$\langle x, y \rangle = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 \right) .$$
   (b) If $\mathbb{F} = \mathbb{C}$, prove that
$$\langle x, y \rangle = \frac{1}{4} \sum_{m=0}^{3} i^m \|x + i^m y\|^2 .$$
   In either case, deduce that two inner products on V are equal if and only if they define the same norm on V.

4. Let $A \in \mathsf{M}_{m \times n}(\mathbb{F})$.
   (a) Prove that for all $x \in \mathbb{F}^n$, $y \in \mathbb{F}^m$, we have
$$\langle Ax, y \rangle_{\mathbb{F}^m} = \langle x, A^* y \rangle_{\mathbb{F}^n} .$$
   (b) Prove that $\mathsf{N}(A) = \operatorname{col}(A^*)^\perp$, as subspaces of $\mathbb{F}^n$ with the standard inner product.