Math 245, Spring 2012
Assignment 5

Due Friday, June 15, in class.

1. Let J_λ be a Jordan block with eigenvalue λ. Let $A_\lambda = e^{tJ_\lambda}$, where $t \in \mathbb{C}$.

   (a) Prove that
   $$(A_0)_{ij} = \begin{cases} t^{j-i}/(j-i)! & \text{if } i \le j \\ 0 & \text{otherwise}. \end{cases}$$

   (b) In general, prove that
   $$(A_\lambda)_{ij} = \begin{cases} e^{\lambda t} t^{j-i}/(j-i)! & \text{if } i \le j \\ 0 & \text{otherwise}. \end{cases}$$

   You may use the following fact: if X and Y are square matrices such that $XY = YX$, then $e^{X+Y} = e^X e^Y$. (*Note:* This is not true if $XY \ne YX$.)

   (c) Suppose $x(t)$ and $y(t)$ are real valued differentiable functions such that
   $$\begin{aligned} x'(t) &= x(t) + 4y(t) \\ y'(t) &= -x(t) + 5y(t). \end{aligned}$$

   If $x(0) = y(0) = 1$, determine $x(t)$ and $y(t)$.

2. A function $f : \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ is said to be *convex* if for all non-negative integers $a \le b \le c$,
   $$(b-a)f(c) + (c-b)f(a) \ge (c-a)f(b).$$

   If T is a linear operator on a finite dimensional complex vector space, prove that $f(m) = \operatorname{rank}(T^m)$ is a convex function.

3. Let V be a finite dimensional vector space over a field $\mathbb{F}$. Let $T \in \mathsf{L}(V)$ be a linear operator. Let W be a T-cyclic subspace, generated by a vector $x \in V$. Suppose that the characteristic polynomial of T_W is equal to $p(t)^d$, where $p(t)$ is a monic irreducible polynomial of degree m in $\mathbb{F}[t]$, and d is a positive integer. (Note that $\dim W = md$, and that the minimial polynomial of T_W is also equal to $p(t)^d$.)

   (a) If $0 \le k \le d$, prove that $\operatorname{im}(p(T_W)^{d-k}) \subset \ker(p(T_W)^k)$. Deduce that
   $$\operatorname{nullity} p(T_W)^{d-k} + \operatorname{nullity} p(T_W)^k \ge md.$$

   (b) Suppose $g(t) \in \mathbb{F}[t]$ is a polynomial of degree $s < md$. Prove that
   $$\{g(T)x\,,\ g(T)Tx\,,\ \ldots\,,\ g(T)T^{md-s-1}x\}$$
   is a linearly independent set. Deduce that $\operatorname{nullity} g(T_W) \le \deg g(t)$.

   (c) Use parts (a) and (b) to prove that $\operatorname{nullity} p(T_W)^k = \min(mk, md)$, for all $k \ge 0$.

(d) Let $q(t)$ be a monic irreducible polynomial. If $q(t) \neq p(t)$, show that $\operatorname{nullity} q(T_W)^k = 0$ for all $k \geq 0$.

(*Hint:* Use the Primary Decomposition Theorem on $\ker(p(T_W)^d q(T_W)^k)$.)

(e) Let $q(t)$ be a monic irreducible polynomial. Show that

$$\frac{2 \operatorname{nullity} q(T_W)^k - \operatorname{nullity} q(T_W)^{k+1} - \operatorname{nullity} q(T_W)^{k-1}}{\deg q(t)} = \begin{cases} 1 & \text{if } f_{T_W}(t) = q(t)^k \\ 0 & \text{otherwise.} \end{cases}$$

(f) Suppose we have a direct sum decomposition $V = W_1 \oplus \cdots \oplus W_s$, where W_i is a T-cyclic subspace and $f_{T_{W_i}}(t)$ is a power of an irreducible polynomial for all i. Let $q(t)$ be a monic irreducible polynomial. Show that

$$\frac{2 \operatorname{nullity} q(T)^k - \operatorname{nullity} q(T)^{k+1} - \operatorname{nullity} q(T)^{k-1}}{\deg q(t)}$$

is a formula for the number of subspaces W_i in the decomposition for which $f_{T_{W_i}}(t) = q(t)^k$.

This exercise shows that one can predict the number of terms of each type in any such decomposition, without actually computing the decomposition! Therefore any two such decompositions—even if they are not identical—will lead to the same canonical form.