Math 245, Spring 2012
Assignment 11

Not to be handed in.

1. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space V over a field $\mathbb{F}$. Let $\{f_1, f_2, \ldots, f_n\}$ be the dual basis.

   (a) Suppose $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. Show that

   $$f_{j_1} \wedge f_{j_2} \wedge \cdots \wedge f_{j_k}(v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}) = \begin{cases} 1 & \text{if } (i_1, i_2, \ldots, i_k) = (j_1, j_2, \ldots, j_k) \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Prove that

   $$\{v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

   is a basis for $\bigwedge^k V$.

2. Let V be a vector space over a field $\mathbb{F}$. Let W, W' be k-dimensional subspaces of V, with bases $\{x_1, x_2, \ldots, x_k\}$ and $\{y_1, y_2, \ldots, y_k\}$ respectively. Prove that

   $$x_1 \wedge x_2 \wedge \cdots \wedge x_k = c\, y_1 \wedge y_2 \wedge \cdots \wedge y_k$$

   for some non-zero scalar $c \in \mathbb{F}$ if and only if $W = W'$.

3. Let $A \in \mathsf{M}_{n\times n}(\mathbb{R})$ be a skew-symmetric matrix, and let $\omega \in \bigwedge^2(\mathbb{R}^n_{\text{col}})$ be the vector

   $$\omega = \sum_{i<j} A_{ij} e_i \wedge e_j\,,$$

   where $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n_{\text{col}}$.

   (a) Prove that $\omega = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n A_{ij} e_i \wedge e_j\,.$

   (b) Let E be an invertible $n \times n$ matrix. Let $B = E^t A E$, and let v_i be the i^{th} column of E^{-1}. Prove that $\omega = \sum_{i<j} B_{ij} v_i \wedge v_j$.

   (c) Using the canonical form for real normal matrices, prove that there exists an orthogonal matrix E such that

   $$E^t A E = \begin{pmatrix} \alpha_1 J & & & & & & \\ & \alpha_2 J & & & & & \\ & & \ddots & & & & \\ & & & \alpha_k J & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}$$

   where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $\alpha_1, \ldots, \alpha_k$ are non-zero real numbers, and $k = \frac{1}{2}\operatorname{rank}(A)$.

(d) Prove that ω can be written in the form

$$x_1 \wedge y_1 + x_2 \wedge y_2 + \cdots + x_k \wedge y_k\,,$$

where $\{x_1, y_1, \ldots, x_k, y_k\}$ is a linearly independent subset of $\mathbb{R}^n$.

(e) If n is even, we define the **Pfaffian** of A to be the unique scalar $\mathrm{Pf}(A)$ with the property that

$$\frac{1}{(n/2)!}\underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n/2} = \mathrm{Pf}(A)\, e_1 \wedge e_2 \wedge \cdots \wedge e_n\,.$$

(i) If A is not invertible, prove that $\mathrm{Pf}(A) = 0$.

(ii) If A is invertible, let E be any invertible matrix such that

$$E^t A E = \begin{pmatrix} \alpha_1 J & & & \\ & \alpha_2 J & & \\ & & \ddots & \\ & & & \alpha_{n/2} J\,. \end{pmatrix}$$

Show that $\mathrm{Pf}(A) = \det(E)^{-1}\, \alpha_1 \cdots \alpha_{n/2}$.

(f) Prove the following facts about Pfaffians:

(i) $\mathrm{Pf}(A^t) = (-1)^{n/2}\, \mathrm{Pf}(A)$.

(ii) $\mathrm{Pf}(\lambda A) = \lambda^{n/2}\, \mathrm{Pf}(A)$, for $\lambda \in \mathbb{R}$.

(iii) $\mathrm{Pf}(A)^2 = \det(A)$.

(iv) $\mathrm{Pf}(B^t A B) = \det(B)\mathrm{Pf}(A)$, for $B \in \mathsf{M}_{n\times n}(\mathbb{R})$.

(*Note:* There are many ways to define the Pfaffian of a skew-symmetric matrix. It is not hard to see from the definition given here that $\mathrm{Pf}(A)$ is a polynomial in the entries of A. There is also a "cofactor expansion"-style formula, from which one can see that this polynomial has integer coefficients. One consequence of this is that the determinant of a skew-symmetric matrix with coefficients in a ring R is always perfect square in R.)