Math 245, Spring 2012
Assignment 10

Due Friday, July 20, in class.

1. Let $x, y \in V$, where V is a finite dimensional vector space over a field $\mathbb{F}$. Prove that $x \otimes y = y \otimes x$ in $V \otimes V$ if and only if x and y are linearly dependent.

2. For each $a \in \mathbb{C}$ let $M_a : \mathbb{C} \to \mathbb{C}$ be the multiplication operator $M_a(b) = ab$. Let V be a finite dimensional vector space over $\mathbb{R}$. Viewing $\mathbb{C}$ as a 2-dimensional vector space over $\mathbb{R}$, consider the tensor product $V \otimes \mathbb{C}$. This is again a vector space over $\mathbb{R}$. We can endow $V \otimes \mathbb{C}$ with the structure of a vector space over $\mathbb{C}$ by defining scalar multiplication as follows:

$$ax = (I_V \otimes M_a)(x) \qquad \text{for } x \in V \otimes \mathbb{C}, \quad a \in \mathbb{C}\,.$$

   Hence $a(v \otimes b) = v \otimes (ab)$ for $v \in V$, $a, b \in \mathbb{C}$.

   (a) Verify that $V \otimes \mathbb{C}$ (with scalar multiplication defined as above) satisfies all the axioms of a vector space over $\mathbb{C}$. This complex vector is called the *complexification* of V, and we denote it $V^{\mathbb{C}}$.

   (b) If $\beta = \{v_1, \ldots, v_n\}$ is a basis for V over $\mathbb{R}$, let $\beta^{\mathbb{C}} = \{v_1 \otimes 1, \ldots, v_n \otimes 1\}$. Prove that $\beta^{\mathbb{C}}$ is a basis for $V^{\mathbb{C}}$.

   (c) For each linear operator $T \in \mathsf{L}(V)$, let $T^{\mathbb{C}} = T \otimes I_{\mathbb{C}}$. Prove that $T^{\mathbb{C}} \in \mathsf{L}(V^{\mathbb{C}})$.
   (*Note:* We already know that $T^{\mathbb{C}}$ is linear over the field of real numbers. The point of this exercise is to show that it's linear over the complex numbers).

   (d) Prove that $[T]_\beta = [T^{\mathbb{C}}]_{\beta^{\mathbb{C}}}$.

   (e) Suppose V is a real inner product space. Prove that there exists a unique inner product $\langle \cdot, \cdot \rangle^{\mathbb{C}}$ on $V^{\mathbb{C}}$ satisfying

$$\langle v \otimes a, w \otimes b \rangle^{\mathbb{C}} = a\bar{b}\langle v, w \rangle$$

   for all $v, w \in V$, $a, b \in \mathbb{C}$. Prove the following: β is orthonormal iff $\beta^{\mathbb{C}}$ is orthonormal; T is normal iff $T^{\mathbb{C}}$ is normal; T is self-adjoint iff $T^{\mathbb{C}}$ is self-adjoint; T is orthogonal iff $T^{\mathbb{C}}$ is unitary.
   (*Note:* You should carefully compare part (e) with Assignment 8, Problem 2. The two problems are closely related, but not *exactly* opposite to each other.)

3. Let V and W be finite dimensional vector spaces over a field $\mathbb{F}$. For $T \in \mathsf{L}(V)$, $U \in \mathsf{L}(W)$, consider the linear operator $T \otimes U \in \mathsf{L}(V \otimes W)$.

   (a) Prove that $\operatorname{rank}(T \otimes U) = \operatorname{rank}(T) \operatorname{rank}(U)$.

   (b) Prove that $\operatorname{tr}(T \otimes U) = \operatorname{tr}(T) \operatorname{tr}(U)$.
   (*Hint*: First prove it for rank-1 linear operators, i.e. $T = vf$ and $U = wg$, where $f \in V^*$, $v \in V$, $g \in W^*$, $w \in W$ are non-zero vectors. Then use the fact that every operator can be written as a sum of rank-1 operators.)