C&O 430/630, Fall 2011
Assignment 4

Due Monday, December 5, in class.

1. *Column insertion* is defined analogously to row insertion, with the roles of rows and columns reversed. Given $T \in \mathrm{SSYT}(\lambda)$ and a number a, we define $a \to T$ to be the tableau obtained by column-merging a into the first column of T; then column merging the output into the second row of T, etc. The subroutine "column-merge", takes as input a column of T, and a number b. If b is greater than every entry in the column, we add b to the bottom of the column and STOP. Otherwise, let c be the topmost entry of the column that is greater than or equal to b. Replace c with b and output c.

   (a) Prove that $a \to T$ is a SSYT.

   (b) Suppose T is a SSYT of shape λ, $a \to T$ has shape λ^+ and $b \to a \to T$ has shape λ^{++}. Prove the following:

   (i) If $a \geq b$ then the box of λ^{++}/λ^+ is strictly right of and weakly above the box of λ^+/λ.

   (ii) If $a < b$ then the box of λ^{++}/λ^+ is strictly below of and weakly left of the box of λ^+/λ.

   (c) Find a variation on the Robinson-Schensted-Knuth correspondence (using column insertion instead of row insertion) to give a combinatorial proof of the identity

$$\prod_{i,j \geq 1} (1 + x_i y_j) = \sum_{\lambda} s_\lambda(x) s_{\lambda^t}(y)\,.$$

   (d) Show that

$$\prod_{i,j \geq 1} (1 + x_i y_j) = \sum_{\lambda} m_\lambda(x) e_\lambda(y)\,.$$

   By applying the fundamental involution to $\Lambda(y)$, deduce that $\omega(s_\lambda) = s_{\lambda^t}$.

2. We say that a skew partition λ/μ is a *ribbon*, if there is at most one path between any two boxes with horizontal and vertical steps; hence λ/μ does not contain a 2×2 square of boxes. The *height* of λ/μ, denoted $\delta(\lambda/\mu)$ is the number of non-empty rows in the diagram λ/μ,

   A ribbon is *connected* if there is unique path between any two boxes with horizontal and vertical steps. A straight-shaped ribbon is called *hook*.

   For example, the partition 41^7 is a hook, $\delta(41^7) = 8$.

   (a) Let λ be a partition of k. Use the Jacobi-Trudi formula to show that

$$\langle m_k, s_\lambda \rangle = \begin{cases} (-1)^{\delta(\lambda)-1} & \text{if } \lambda \text{ is a hook} \\ 0 & \text{otherwise}\,. \end{cases}$$

   (b) Prove that

$$p_k = m_k = \sum (-1)^{\delta(\nu)-1} s_\nu\,,$$

   where the sum is taken over all hooks ν of size k.

(c) Let μ be an arbitrary partition, and let ν be a hook. Prove that if if λ/μ is not a ribbon, then $\#\mathcal{LR}_{\lambda/\mu;\nu} = 0$.

(d) Let ν be a hook, and let λ/μ be a ribbon, with $|\lambda/\mu| = |\nu|$. Suppose that λ/μ has exactly d connected components. Prove that

$$\#\mathcal{LR}_{\lambda/\mu;\nu} = \binom{d-1}{\delta(\lambda/\mu) - \delta(\nu)} .$$

(e) Prove that

$$s_\mu p_k = \sum (-1)^{\delta(\lambda/\mu)-1} s_\lambda ,$$

where the sum is taken over all λ such that λ/μ is a connected ribbon of size k.

(f) Now, let $\lambda, \mu \vdash n$ be partitions of n. Let μ' be obtained by removing a part of size k from μ. Prove that

$$\langle s_\lambda, p_\mu \rangle = \sum (-1)^{\delta(\lambda/\lambda')-1} \langle s_{\lambda'}, p_{\mu'} \rangle ,$$

where the sum is taken over all λ' such that λ/λ' is a connected ribbon of size k.

(g) For each partition $\mu \vdash 6$, compute $\langle s_{321}, p_\mu \rangle$. (Hint: For most μ the answer is 0.)

3. (a) Prove that

$$\mathrm{ex}(m_\lambda) = \begin{cases} \frac{1}{n!} & \text{if } \lambda = 1^n,\ n \geq 1 \\ 0 & \text{otherwise}. \end{cases}$$

(b) Let λ/μ be a skew partition. Prove that

$$\mathrm{ex}(s_{\lambda/\mu}) = \frac{\#\mathrm{SYT}(\lambda/\mu)}{|\lambda/\mu|!} .$$

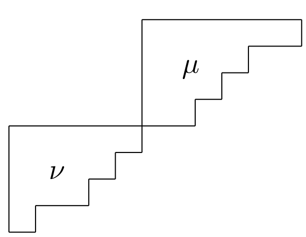
(c) The Jacobi-Trudi formula for skew Schur functions asserts that if λ has at most m parts, and $\mu \subseteq \lambda$, then

$$s_{\lambda/\mu} = \left(h_{\lambda_i - \mu_j - i + j}\right)_{i,j=1,\dots,m} .$$

(The proof is a straightforward extension of non-skew case. See Stanley's "Enumerative Combinatorics, Vol. 2." for details.)

By considering $\mathrm{ex}(s_{\lambda/\mu})$ in the case where λ/μ is a connected ribbon, give an alternate proof of the result from Assignment 2, Problem 6(b).

4. If $\mu = (\mu_1, \dots, \mu_m)$ and $\nu = (\nu_1, \dots, \nu_n)$ are partitions, let $\mu * \nu$ denote the skew partition



$$= \quad (\mu_1{+}\nu_1, \mu_2{+}\nu_1, \dots, \mu_m{+}\nu_1, \nu_1, \nu_2, \dots, \nu_n)/\nu_1^m .$$

(a) Prove that $c^\lambda_{\mu,\nu}$ is equal to the number of Littlewood-Richardson tableaux of shape $\mu * \nu$ and content λ.

(b) Describe a bijection between $\mathcal{LR}_{\lambda/\mu;\nu}$ and $\mathcal{LR}_{\mu*\nu;\lambda}$. (No proof is required.)

Appendix A. Acknowledgements

Collaboration and discussion are an important part of the learning process and academic life in general. It is equally important to acknowledge the influence that others have had in our own work.

Please list the names of everyone (enrolled in the course or otherwise) with whom you discussed any homework problems from this course. Briefly describe the nature and extent of your discussions; e.g. did you work together on solutions to several problems, or just exchange the occasional hint? If you consulted any reference materials that are not listed on the course webpage, please take this opportunity to acknowledge these as well.

If you did not discuss homework with anyone, or use any outside references for the homework, please affirm this.