C&O 430/630, Fall 2011
Assignment 3

Due Monday, November 21, in class.

1. A *tournament* is an orientation of the complete graph. Let $\mathcal{T}_n$ be the set of all tournaments on vertices $\{1, \dots, n\}$. For $\tau \in \mathcal{T}_n$, let $\mathrm{wt}_j(\tau)$ be the out-degree of vertex j, and let $\mathrm{ind}(\tau)$ be the number of oriented edges $i \to j$ with $i > j$.

   (a) If $T(x_1, \dots, x_n; u) = \sum_{\tau \in \mathcal{T}_n} u^{\mathrm{ind}(\tau)} x_1^{\mathrm{wt}_1(\tau)} \cdots x_n^{\mathrm{wt}_n(\tau)}$ prove that

   $$T(x_1, \dots, x_n; u) = \prod_{1 \le i < j \le n} (x_i + u x_j) \; .$$

   (b) Let $V(x_1, \dots, x_n) = \det \left(x_i^{n-j} \right)_{i,j=1,\dots,n}$, the Vandermonde determinant. Define a graph structure on $\mathcal{T}_n$ by joining σ and τ if σ is obtained from τ by reversing the edges in an directed cycle of length 3. Prove that every component of $\mathcal{T}_n$ is regular and bipartite. Hence, give a combinatorial proof that

   $$V(x_1, \dots, x_n) = \prod_{1 \le i < j \le n} (x_i - x_j) \; .$$

   (A regular bipartite graph structure generalizes the notion of a sign reversing involution, which is a 1-regular bipartite graph structure.)

2. If T is a straight shaped SSYT, define a new tableau $\alpha(T)$ as follows. Let x be the entry in the lower left corner of T. Let T^- be the tableau obtained by deleting this entry x from T, and realigning the bottom row. Let $\alpha(T) = T^- \leftarrow x$.

   Example: If $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$, then $x = 3$, $T^- = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}$, and $\alpha(T) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$.

   (a) For $a \ge 1$, prove that $R_a(\alpha(T)) = \alpha(R_a(T))$. (You may use, without proof, the results of Exercise 4.4.4 and Theorem 4.5.4 from the notes.)

   (b) Prove that the sequence $T, \alpha(T), \alpha(\alpha(T)), \dots$ contains a tableau with only one row. (Hint: In this sequence, what happens to the entries in the top row?)

   (c) Prove that the crystal reflection operators satisfy the braid relations: i.e.

   $$R_a \circ R_b(T) = R_b \circ R_a(T) \quad \text{if } |a - b| \ge 2 \qquad \text{and} \qquad R_a \circ R_{a+1} \circ R_a(T) = R_{a+1} \circ R_a \circ R_{a+1}(T) \, .$$

3. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ be a partition of n with d parts.

   (a) Let $\mathcal{P}$ be the set of all functions $f : [d] \to \mathbb{Z}_{>0}$. Define a weight function $\mathrm{wt}^\lambda(f) = (\mathrm{wt}_1^\lambda(f), \mathrm{wt}_2^\lambda(f), \dots)$ on $\mathcal{P}$ by

   $$\mathrm{wt}_i^\lambda(f) = \sum_{j : f(j) = i} \lambda_j \, .$$

   Prove that the generating function for $\mathcal{P}$ with respect to wt^λ is $p_\lambda(x_1, x_2, \dots)$.

(b) Suppose $\mathrm{wt}^\lambda(f)$ is a partition, i.e. $\mathrm{wt}_1^\lambda(f) \geq \mathrm{wt}_2^\lambda(f) \geq \dots$. Prove that $\mathrm{wt}^\lambda(f) \geq_{\mathrm{lex}} \lambda$, and determine the number of $f \in \mathcal{P}$ for which equality occurs.

(c) Prove that there exist integers $M_{\lambda\mu}$ such that

$$p_\lambda = \sum_{\mu \geq_{\mathrm{lex}} \lambda} M_{\lambda\mu} m_\mu \,,$$

where $M_{\lambda\lambda} > 0$. Hence prove that $\{p_\lambda \mid \lambda \vdash n\}$ is a basis for Λ_n.

(d) Let $\mathcal{C}_\lambda$ be the set of permutations in S_n with cycles of sizes $\lambda_1, \lambda_2, \dots, \lambda_d$. Prove that $|\mathcal{C}_\lambda| = \frac{n!}{z(\lambda)}$.

(e) Let $\pi \in \mathcal{C}_\lambda$ be a permutation, and consider the set $\mathcal{F}_\pi$ of all functions $f : [n] \to \mathbb{Z}_{>0}$ such that $f \circ \pi = f$. Define a weight function $\mathrm{wt}(f) = (\mathrm{wt}_1(f), \mathrm{wt}_2(f), \dots)$ on $\mathcal{F}_\pi$ where

$$\mathrm{wt}_i(f) = \#\{k \mid f(k) = i\}\,.$$

Prove that the generating function for $\mathcal{F}_\pi$ with respect to wt is $p_\lambda(x_1, x_2, \dots)$.

(f) Give a combinatorial proof of the identity

$$\prod_{i,j\geq 1} (1 - x_i y_j)^{-1} = \sum_\lambda \frac{p_\lambda(x) p_\lambda(y)}{z(\lambda)}\,,$$

by finding a suitable bijection between the set of all pairs

- $(\{(a_1, b_1), \dots, (a_n, b_n)\}, \sigma)$, where $\{(a_1, b_1), \dots, (a_n, b_n)\}$ is a multiset of ordered pairs of positive integers and $\sigma \in S_n$,

and the set of all triples

- (f, g, π), where $\pi \in S_n$ and $f, g \in \mathcal{F}_\pi$.

4. Prove that

$$h_n = \frac{1}{n!} \det \begin{pmatrix} p_1 & -1 & 0 & \dots & 0 \\ p_2 & p_1 & -2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_{n-2} & p_{n-3} & \dots & 1-n \\ p_n & p_{n-1} & p_{n-2} & \dots & p_1 \end{pmatrix} \qquad \text{for } n \geq 1\,.$$

5. Suppose that $x_1^k + x_2^k + \dots x_n^k = k$, for $k = 1, \dots, n$. Evaluate $x_1^{n+1} + \dots + x_n^{n+1}$.
(Hint: Use (5.3.3) from the notes.)