C&O 430/630, Fall 2011
Assignment 2

Due Wednesday, November 2, in class.

1. (a) Let $\mathcal{F}$ be the species of endofunctions: $\mathcal{F}_X$ is the set of all functions $\alpha : X \to X$, and $f_* = f \circ \alpha \circ f^{-1}$ for any bijection $f : X \to Y$. By viewing an endofunction as a directed graph, find a natural equivalence $\mathcal{F} \simeq \mathcal{S}[\mathcal{T}^\bullet]$, and hence prove the identity

$$n^n = \sum_{\substack{k, i_1, \ldots, i_k \geq 1 \\ i_1 + \cdots + i_k = n}} \binom{n}{i_1 \; \ldots \; i_k} i_1^{i_1 - 1} \ldots i_k^{i_k - 1} ,$$

where $\binom{n}{i_1 \; \ldots \; i_k} = \frac{n!}{i_1! \cdots i_k!}$.

(b) Let $\hat{\mathcal{F}}_X \subset \mathcal{F}_X$ be the set of all fixed point free endofunctions $\alpha : X \to X$, i.e. all endofunctions such that $\alpha(x) \neq x$. Find a natural equivalence $\mathcal{F} \simeq \hat{\mathcal{F}} * \mathcal{T}'$, and hence prove the identity

$$n^n = \sum_{k=0}^{n} \binom{n}{k} (k+1)^{k-1} (n-k-1)^{n-k} .$$

(A sketch of proof is okay — you don't need to prove formally that the maps are *natural* equivalences.)

2. (a) Let $\mathcal{P}$ be the species of set partitions, where the weight of a set partition $\{S_1, \ldots, S_k\}$ is defined to be k. Show that the mixed generating function for $\mathcal{P}$ is

$$P(x; t) = \sum_{n, k \geq 0} S(n, k) t^k \frac{x^n}{n!} = e^{t(e^x - 1)} .$$

By convention, the empty set is a valid set partition of the empty set, hence $S(0, 0) = 1$. The numbers $S(n, k)$ are called Stirling numbers of the second kind.

(b) Show that $S(n, k)$ satisfy the recurrence relation

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

for $n, k \geq 1$, with initial conditions $S(0, 0) = 1$, $S(n, 0) = S(0, k) = 0$ for $n, k \geq 1$.

(c) Show that

$$x^n = \sum_{k=0}^{n} S(n, k) \, x(x-1)(x-2) \ldots (x-k+1) .$$

(d) Prove that the signed Stirling numbers of the first kind, $\tilde{s}(n, k) = n![x^n t^k](1+x)^t$, define a matrix that is inverse to $S(n, k)$, i.e. prove that

$$\sum_{m \geq 0} \tilde{s}(n, m) S(m, k) = \sum_{m \geq 0} S(n, m) \tilde{s}(m, k) = \delta_{n,k} .$$

3. (a) Find a formula for the number of trees in $\mathcal{T}_{[n]}$ such that vertex n is a leaf.

   (b) Let $t_{n,k}$ be the number of trees in $\mathcal{T}_{[n]}$ with exactly k leaves. Let $T(x;u)$ be the mixed generating function for the species of rooted trees, weighted by the number of vertices of updegree 0. Determine $T(x;u)$ and show that $t_{n,k} = \frac{n!}{k}[x^{n-1}u^{k-1}]T(x;u)$.

   (c) Prove that

$$t_{n,k} = S(n-2, n-k)\, n(n-1)\dots(k+1)\,,$$

   where $S(n,k)$ is the Stirling number of the second kind. Hence show that $S(n,k)$ is the number of trees with leaves labelled $1,\dots,n{-}k{+}2$, and k unlabelled nonleaves.

4. (a) *Frustration* is a solitaire card game. The player calls out the sequence:

   "Ace", "Two", "Three", ... , "Jack", "Queen", "King", "Ace", "Two", etc.

   With each call, he simultaneously flips over a card from a shuffled deck. The object of the game is to make it through the entire deck without ever calling out the rank of the card being flipped over. Determine the probability of winning frustration.

   (b) Norah finds this too banal, and so prefers to play a variant called *the π game.* In this version the player calls out (yep, you guessed it):

   "Three", "Point", "One", "Four", "One", "Five", "Nine", etc.

   Aces count as "One", but flipping over a Ten, Jack, Queen, or King is always safe, as is calling out "Point" or "Zero". Determine the probability of winning the π game.

   Your answers may in be any form, but should be tractably computable.

5. In this exercise, we identify a permutation π of $[n]$ with the linear order $\pi(1), \pi(2), \dots, \pi(n)$. Let $a_1, \dots, a_k$ be a sequence of distinct elements from a totally ordered set. The *pattern* of $a_1, \dots, a_k$ is defined to be the permutation σ of $[k]$ such that $a_{\sigma^{-1}(1)} < \cdots < a_{\sigma^{-1}(k)}$. (For example, the pattern of $4, 6, 1, 9$ is 2314.)

   (a) Consider the linear species $\mathcal{A}$ of linear orders with marked substrings of pattern 213. (For example,

$$3, 1, \boxed{8{,}5{,}9}, 7, \boxed{4{,}2{,}6} \in \mathcal{A}_{[9]}\,.$$

   Here there are two marked substrings of pattern 213. Note that marked substrings could overlap, though in this example they do not.) Defined the weight of an $\mathcal{A}$-structure to be the number of marked substrings. Compute the mixed generating function for $\mathcal{A}$. Hence, determine the number of linear orders of $[n]$ that have no substring of pattern 213.

   (b) Using a similar approach, determine the number of linear orders of $[n]$ that have no substring of pattern 123.

   Express your answers as a coefficient of a generating function. *Note:* The answer to part (a) involves a formal integral without a closed form. Don't let this bother you.

6. (a) We say that a permutation σ of $[n]$ has a *descent* at position i if $\sigma(i) > \sigma(i+1)$. Let $a_1, \ldots, a_m$ be integers, with $1 \leq a_1 < \cdots < a_m \leq n-1$. Use inclusion-exclusion to show that the number of permutations of $\{1, \ldots, n\}$ with descents at positions $a_1, \ldots, a_m$ (and nowhere else) is

$$\sum_{\substack{j \geq 0 \\ 1 \leq i_1 < \cdots < i_j \leq m}} (-1)^{m-j} \frac{n!}{\prod_{\ell=1}^{j+1} (a_{i_\ell} - a_{i_{\ell-1}})!},$$

where, by convention, $a_{i_0} = 0$ and $a_{i_{j+1}} = n$ in the summation.

(b) Let $(A_{ij})_{i,j=0,\ldots,m}$ be the matrix

$$A_{ij} = \begin{cases} \frac{1}{(a_{j+1} - a_i)!} & \text{if } j+1 \geq i \\ 0 & \text{otherwise,} \end{cases}$$

where $a_0 = 0$ and $a_{m+1} = n$. Prove that the summation in part (a) is equal to $n! \det(A)$.