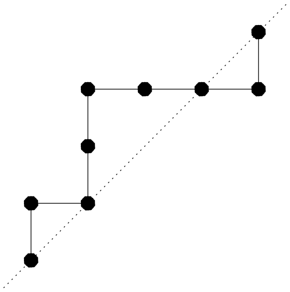
C&O 430/630, Fall 2011
Assignment 1

Due Friday, October 14, in class.

1. Use Hensel's Lemma to prove the following.
   (a) Suppose that $A(x), B(x) \in \mathbb{Q}[[x]]$, are formal power series such that $A(0) = B(0) = 1$ and $A(x)^{B(x)} = B(x)^{A(x)}$. Prove that $A(x) = B(x)$.
   (b) Let $F(t, x) \in \mathbb{C}[[t]][x]$ be a polynomial of degree n in variable x (with coefficients in $\mathbb{C}[[t]]$). Suppose that $F(0, x) \in \mathbb{C}[x]$ also has degree n and has no repeated roots. Prove that there exist formal power series $c(t), r_1(t), \ldots, r_n(t) \in \mathbb{C}[[t]]$ such that
   $$F(t, x) = c(t)(x - r_1(t))(x - r_2(t)) \cdots (x - r_n(t)) .$$
   Give examples to show that this may be false if either hypothesis about $F(0, x)$ is omitted.

2. Let R be an integral domain, and let $\Phi(x) \in R[[x]]_+$ be a formal power series.
   (a) Prove that $\Phi(x)$ has a two-sided compositional inverse $\Psi(x) = \Phi^{[-1]}(x) \in R[[x]]_+$ if and only if $[x]\Phi(x)$ is invertible in R.
   (b) Suppose $\mathbb{Q} \subset R$, and $\Psi(x) = \Phi^{[-1]}(x)$. Let A and B be the infinite matrices
   $$A_{ij} = [x^i]\Phi(x)^j \qquad\qquad B_{ij} = [x^i]\Psi(x)^j , \qquad i, j \in \mathbb{Z} .$$
   Show that A and B are lower triangular matrices, i.e. $A_{ij} = B_{ij} = 0$ if $i < j$. Also, prove that $iA_{i,j} = jB_{-j,-i}$ for all $i, j \in \mathbb{Z}$.
   (c) Prove that A and B are inverse matrices, i.e. show that that $(AB)_{ik} = (BA)_{ik} = \delta_{ik}$, where
   $$(AB)_{ik} = \sum_{j\in\mathbb{Z}} A_{ij}B_{jk} = \sum_{j=k}^{i} A_{ij}B_{jk} .$$
   (d) If $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are sequences such that
   $$b_i = \sum_{j=1}^{i} \frac{j \cdot i^{i-j-1}}{(i-j)!}\, a_j ,$$
   prove that
   $$a_i = \sum_{j=1}^{i} \frac{(-j)^{i-j}}{(i-j)!}\, b_j .$$

3. A lattice path with steps $(0, 1)$ and $(1, 0)$ is a finite sequence $p = (p_0, p_1, \ldots, p_m)$ of points in $\mathbb{Z}^2$ such that $p_i - p_{i-1} \in \{(0, 1), (1, 0)\}$. Let $\mathcal{P}$ denote the set of all such lattice paths that start at $p_0 = (0, 0)$ and end at $p_{2n} = (n, n)$ for some $n \geq 0$. We define two weight functions on $\mathcal{P}$: let $\mathrm{wt}_1(p)$ be the number of horizontal steps in p above the line $y = x$; let $\mathrm{wt}_2(p)$ be the number of horizontal steps below the line $y = x$.



For example, the lattice path p above has $\mathrm{wt}_1(p) = 3$ and $\mathrm{wt}_2(p) = 1$.

Write the generating function for $\mathcal{P}$ (with respect to wt_1 and wt_2) as

$$P(x,y) = \sum_{n \geq k \geq 0} p_{n,k}\, x^{n-k} y^k \,.$$

Let $\mathcal{D}$ be the subset of $\mathcal{P}$ of paths with no steps below the line $y = x$, and let $D(x)$ denote the generating function for $\mathcal{D}$, with respect to wt_1.

(a) Show that

$$P(x,y) = \frac{1}{1 - xD(x) - yD(y)} \,.$$

(b) Use the functional equation $D(x) = 1 + xD(x)^2$ to rewrite this expression in a way that shows that $p_{n,k}$ is independent of $k \in \{0, 1, \ldots, n\}$. Hence, show that $p_{n,k} = \frac{1}{n+1}\binom{2n}{n}$ for all k. (Hint: Work backwards!)

4. Let $\mathcal{A}$ denote the set of strings $\sigma = \sigma_1\sigma_2 \ldots \sigma_{\mathrm{length}(\sigma)}$ on alphabet $1, 2, 3, ..., m$. (If $m = 5$, some examples are 13325155 and 241444.) Define weight functions on $\mathcal{A}$, $\mathrm{wt}_1, \ldots, \mathrm{wt}_m$ by $\mathrm{wt}_i(\sigma) =$ number of i's in σ.

   (a) Determine the generating function

   $$A(x_1, \ldots, x_m) = \sum_{\sigma \in \mathcal{A}} x_1^{\mathrm{wt}_1(\sigma)} \ldots x_m^{\mathrm{wt}_m(\sigma)} \,.$$

   (b) Let $\mathcal{B} \subset \mathcal{A}$ be the set of strings σ for which $\sigma_i \neq \sigma_{i+1}$ for $i < \mathrm{length}(\sigma)$. (For example, $1423415142 \in \mathcal{B}$, but $214431 \notin \mathcal{B}$ since it has two consecutive 4's.) Consider the generating function

   $$B(x_1, \ldots, x_m) = \sum_{\sigma \in \mathcal{B}} x_1^{\mathrm{wt}_1(\sigma)} \ldots x_m^{\mathrm{wt}_m(\sigma)} \,.$$

   Find a function $f(x)$ such that $B(f(x_1), \ldots, f(x_n)) = A(x_1, \ldots, x_n)$, and hence determine $B(x_1, \ldots, x_m)$.

   (c) Let $\mathcal{C} \subset \mathcal{A}$ be the set of strings for which $\sigma_i + 1 \neq \sigma_{i+1}$ for $i < \mathrm{length}(\sigma)$. (For example, $1133552 \in \mathcal{C}$, but $1134225 \notin \mathcal{C}$.) Determine the generating function

   $$C(x_1, \ldots, x_m) = \sum_{\sigma \in \mathcal{C}} x_1^{\mathrm{wt}_1(\sigma)} \ldots x_m^{\mathrm{wt}_m(\sigma)} \,.$$

   Your answers to parts (a)–(c) should be expressed as (unsimplified) rational functions in $x_1, \ldots, x_m$.