27 November 18

In order to talk about representation theory of S_n, we need some general theory which will apply to any finite group G. A (finite dimensional) *representation* of G consists of a finite dimensional complex vector space V and a group homomorphism $\rho_V : G \to \mathrm{GL}(V)$. (A homomorphism is a map satisfying $\rho(1) = I_V$, and $\rho_V(gh) = \rho_V(g)\rho_V(h)$ for all $g, h \in G$; $\mathrm{GL}(V)$ is the set of invertible linear operators on V.) It is common practice to abuse terminology and call V alone (without reference to ρ_V) the representation. It is also common to write gx to mean $\rho_V(g)x$, where $x \in V$, $g \in G$.

If V is a representation of G. A subspace $W \subset V$ is called a subrepresentation if W is $\rho_V(g)$-invariant (i.e. $\rho_V(g)W \subset W$) for all $g \in G$. A subrepresentation $W \subset V$ is a representation of G (where $\rho_W(g) = \rho_V(g)|_W$ for all $g \in G$).

Before we get too deeply into the theory, here are some example of representations and subrepresentations for the group S_n.

Example 27.1. Let $V = \mathbb{C}^n$ with standard basis $\{e_1, \ldots, e_n\}$, and for $\pi \in S_n$, let $\rho(\pi)$ be the permutation matrix defined by $\rho(\pi)e_i = e_{\pi(i)}$. Then V is a representation of S_n.

The subspace T spanned by $\{(1, \ldots, 1)\}$ is a subrepresentation. This is a "trivial" representation, in that $\pi x = x$ for all $x \in T$, $\pi \in S_n$.

The subspace $\tilde{V} = \{(x_1, \ldots, x_n) \mid \sum x_i = 0\}$ is also a subrepresentation. Notice that T and $\tilde{V}$ are complementary to each other, in the sense that $V = T \oplus \tilde{V}$.

Example 27.2. Let $A = \mathbb{C}^1$, and for $\pi \in S_n$ define $\rho_A(\pi) = \mathrm{sgn}(\pi)$, thought of as a 1×1 matrix. Since $\mathrm{sgn}(\pi\sigma) = \mathrm{sgn}(\pi)\mathrm{sgn}(\sigma)$ this is a representation. A is called the alternating representation of S_n and is different from the one dimensional representation T.

Example 27.3. Let $G = S_n$ and let $V = \mathbb{C}[x_1, \ldots, x_n]_k$ be the vector space of homogeneous polynomials of degree k. Let $\rho_V(\pi)f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$. Then V is a representation of S_n.

The subspace $\Lambda_k^{(n)}$ of symmetric polynomials of degree k in n variables, is a subrepresentation of V. Another example of a subrepresentation is the span of the monomials $x_1^k, \ldots, x_n^k$.

Example 27.4. Let $\mathcal{A}$ be a species. Define $\mathbb{C}\mathcal{A}_n$ to be a vector space whose basis is $\mathcal{A}_{[n]}$. Thus the elements of $\mathbb{C}\mathcal{A}_n$ are formal linear combinations of elements of $\mathcal{A}$-structures on $\{1, \ldots, n\}$. If $\pi \in S_n$ we define $\rho_{\mathbb{C}\mathcal{A}_n}$ by setting $\rho_{\mathbb{C}\mathcal{A}_n}(\pi)(A) = \pi_*(A)$ for $A \in \mathcal{A}_{[n]}$, and extending this to a linear transformation. This makes $\mathbb{C}\mathcal{A}_n$ a representation of S_n.

For example, for the species $\mathcal{E}^\bullet$, an $\mathcal{E}^\bullet$-structure on $[n]$ is a pair $e_i = ([n], i)$ where $i \in [n]$. Thus $\mathbb{C}\mathcal{E}^\bullet$ is the n-dimensional vector space spanned by $\{e_1, \ldots, e_n\}$. If $\pi \in S_n$ then $\pi e_i = \pi_*(e_i) = e_{\pi(i)}$. So $\mathbb{C}\mathcal{E}^\bullet$ is the same as the representation V in Example 27.1. The representation $\tilde{V}$ in Example 27.1, however is not of this form; neither is A in Example 27.2

Example 27.5. We'll be most interested in a special case of this last example, where the species is

$$\mathcal{M}^\lambda = \mathcal{E}_{\lambda_1} * \mathcal{E}_{\lambda_2} * \cdots * \mathcal{E}_{\lambda_d}.$$

Here, $\lambda = (\lambda_1, \ldots, \lambda_d)$ is a partition of n. In this case the representation $\mathbb{C}\mathcal{M}^\lambda_{[n]}$ is given the name M^λ. An element of $\mathcal{M}^\lambda_{[n]}$ is a set composition $(S_1, \ldots, S_d)$ of $[n]$ where $\#S_i = \lambda_i$.

We will represent such an element by a tableau-like diagram called a *tabloid*, wherein the elements of S_i are listed in i^{th} row. Often they are written in increasing numerical order, but that is simply convention; to indicate that the order of the elements within each row doesn't matter we leave out the vertical line separating the boxes. For example,

$$\mathfrak{U} = \begin{array}{llll} \hline 3 & 5 & 6 & 9 \\ \hline 1 & 4 & 8 & \\ \cline{1-3} 2 & & & \\ \cline{1-1} 7 & & & \\ \cline{1-1} \end{array}$$

is a tabloid of shape 4311; if $\pi = 283941756$, then

$$\pi\mathfrak{U} = \begin{array}{llll} \hline 3 & 4 & 1 & 6 \\ \hline 2 & 9 & 5 & \\ \cline{1-3} 8 & & & \\ \cline{1-1} 7 & & & \\ \cline{1-1} \end{array} = \begin{array}{llll} \hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 & 9 & \\ \cline{1-3} 8 & & & \\ \cline{1-1} 7 & & & \\ \cline{1-1} \end{array} .$$

Proposition 27.6. *Let V be a representation of G. There is a Hermitian postive definite inner product $H_V(\cdot,\cdot)$ on V such that for all $x, y \in V$, $g \in G$,*

$$H_V(gx, gy) = H_V(x, y) .$$

Such an inner product is called a G-invariant inner product.

Proof. Let $\tilde{H}(x, y)$ be any positive definite inner product on V. It is easy to check that $H_V(x, y) = \sum_{g\in G} \tilde{H}(gx, gy)$ has the desired properties. □

Proposition 27.7. *Let V be a representation of G. If $W \subset V$ is a subrepresentation then there is another subrepresentation $W' \subset V$ such that $V = W' \oplus W$.*

Proof. Define $W' = W^{\perp}$, the orthogonal complement to W under a G-invariant inner product. Then $V = W \oplus W'$. Then if $y \in W'$ then $H_V(y, g^{-1}w) = H_V(gy, w)$ for all $w \in W$. so $gy \in W'$; hence W' is a subrepresentation. □

We call a representation V reducible if it has a proper non-zero subrepresentation W, $\{0\} \neq W \subsetneq V$. We call V *irreducible* if it is not reducible. The representations T and $\tilde{V}$ from Example 27.1, and A from Example 27.2 are examples of irreducible representations. The phrase "irreducible representation" is often abbreviated as "irrep".

Corollary 27.8. *Every representation of V can be decomposed as $V = V_1 \oplus \cdots \oplus V_k$, where V_i is an irreducible representation.*