COMBINATORIAL HOPF ALGEBRAS LECTURE 9 SUMMARY

WINTER 2020

Summary

Today we built the Faà di Bruno Hopf algebra as an incidence Hopf algebra.

First, a set partition of a set S is a collection $\sigma = \{B_1, \dots, B_k\}$ of disjoint nonempty subsets of S, called blocks, with union S. We say σ refines τ if each part of σ is contained in some part of τ, and write $\sigma \leq \tau$. This gives a poset. Write Π_n for this poset of set partitions of $\{1, \dots, n\}$.

We drew an example of an interval in this poset and noticed its product structure. In general for $[\sigma, \tau]$ if λ_i is the number of blocks of τ that are broken into exactly i blocks of σ then

$$[\sigma, \tau] = \Pi_1^{\lambda_1} \times \Pi_2^{\lambda_2} \times \cdots \times \Pi_n^{\lambda_n}$$

and two intervals are isomorphic if an only if their λ_is agree.

We can form an incidence Hopf algebra with the set of intervals of Π_ns and with poset isomorphism as the relation. We discussed the required hypotheses, all of which hold in this case.

To calculate the coproduct write $a_n = [\Pi_n]$. Then

$$\begin{aligned}
\Delta(a_n) &= \sum_{\substack{\sigma \text{ set partition} \\ \text{of } \{1,2,\dots,n\}}} [\{\{1\},\{2\},\dots,\{n\}\}, \sigma] \otimes [\sigma, \{\{1,2,\dots,n\}\}] \\
&= \sum_{k=1}^{n} \sum_{\substack{\sigma \text{ with} \\ k \text{ parts}}} a_1^{\lambda_1} \cdots a_n^{\lambda_n} \otimes a_k \\
&= \sum_{k=1}^{n} \sum_{\substack{\lambda_1+\lambda_2+\cdots+\lambda_n=k \\ 1\lambda_1+2\lambda_2+\cdots+n\lambda_n=n}} \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_n!(1!)^{\lambda_1}(2!)^{\lambda_2}\cdots(n!)^{\lambda_n}} a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \otimes \lambda_k
\end{aligned}$$

where λ_i in the second line is the number of parts of σ of size i. The coefficient in the last line is the number of set partitions with the given λ sequence. One way to see this is to note that if the set partition were ordered then the answer would be given by a multinomial coefficient, specifically $\binom{n}{1,\dots,1,2,\dots,2,\dots,n,\dots,n}$ where there are λ_1 1s, λ_2 2s etc. But the set partition is not ordered, so we need to divide out by the number of permutations of the parts of the same size, namely $\lambda_1!\lambda_2!\cdots\lambda_n!$.

This is the Faà di Bruno Hopf algebra. How does usual Faà di Bruno come in? Let f, g be formal diffeomorphisms, that is formal power series with 0 constant term. Write them exponentially $f(x) = \sum_{n\geq 1} \frac{f_n}{n!} x^n$, $g(x) = \sum_{n\geq 1} \frac{g_n}{n!} x^n$. What is $[x^n](f \circ g)$? The answer which you can work out in various ways is the Faà di Bruno formula

$$[x^n](f\circ g) = \sum_{k=1}^{n} \frac{f_k}{k!} \sum_{\lambda_1+\lambda_2+\cdots+\lambda_n=k \, 1\lambda_1+2\lambda_2+\cdots+n\lambda_n=n} \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_n!(1!)^{\lambda_1}(2!)^{\lambda_2}\cdots(n!)^{\lambda_n}} g_1^{\lambda_1} g_2^{\lambda_2} \cdots g_n^{\lambda_n}$$

We could define a Hopf algebra on the dual of the space of formal diffeomorphisms. The $[x^n]$ operator is in this dual, so let $a_n = [x^n]$ and consider a Hopf algebra with algebra structure given by $K[a_1, a_2, \ldots]$. We want a coproduct so that $a_n(f \circ g) = m(\Delta a_n)(g \otimes f)$, which is exactly what the coproduct defined above does.

References

Hector Figueroa, Jose M. Gracia-Bondia, Joseph C. Varilly, *Faa di Bruno Hopf algebras*, `https://arxiv.org/abs/math/0508337`