COMBINATORIAL HOPF ALGEBRAS LECTURE 8 SUMMARY

WINTER 2020

Summary

Recall the incidence coalgebra $C(\mathcal{P})$ from last time. The dual $C(\mathcal{P})^*$ is called the *incidence algebra*. (This is the direction of dual that always works so we don't need a graded dual.) If you work out what the multiplication is, you see that this is in fact the convolution algebra from the coalgebra $C(\mathcal{P})$ to K as an algebra over itself.

We gave two examples of incidence algebras. First if $P = (\mathbb{Z}_{\geq 0}, \geq)$ then the intervals are $[m, n]$ for $m \leq n$. Consider the incidence algebra of the intervals of P. An element of this incidence algebra is functions assigning a scalar to each pair $m \leq n$, so this is infinite upper triangular matrices. The multiplication, if you work it out, turns out to be matrix product.

Second if we take the reduced incidence algebra of the above example, then every interval $[m, n] \sim [0, n-m]$, so we can take representatives of the equivalence classes to be the intervals $[0, i]$ for $i \geq 0$. This means that an element of the incidence algebra is a function on $\mathbb{Z}_{\geq 0}$, that is a sequence, or equivalently a formal power series. The multiplication, if you work it out, turns out to be formal power series multiplication.

Now this is not the algebra structure we want for our incidence Hopf algebras. They will be based on the incidence coalgebras with a different multiplication.

Definition 1. *Let P_1 and P_2 be posets. The* direct product *of P_1 and P_2, denotes $P_1 \times P_2$ is a poset whose underlying set is the cartesian product and where $(x_1, x_2) \leq (y_1, y_2)$ iff $x_1 \leq y_1$ in P_1 and $x_2 \leq y_2$ in P_2.*

Cartesian product of sets is associative and we'll take the isomorphisms used in this associativity as automatic so that direct product is also associative.

Definition 2. *A* hereditary family *of intervals is an interval closed family $\mathcal{P}$ which is also closed under direct product.*

This is the kind of family we want for our Hopf algebras.

We want to use direct product as the product of our Hopf algebra. It has two problems, it isn't compatible with the order compatible relations, and it doesn't have a unit necessarily. We will fix both those problems by additional restrictions on what is an allowable relation.

Definition 3. *Suppose $\mathcal{P}$ is a hereditary family and $\sim$ is an order compatible equivalence relation on $\mathcal{P}$ such that*

- *if $P \sim Q$ then $P \times R \sim Q \times R$ and $R \times P \sim R \times Q$ for all $R \in \mathcal{P}$ (so $\sim$ is a semigroup congruence)*
- *if $|Q| = 1$ then $P \times Q \sim P \sim Q \times P$. (so $\sim$ is reduced),*

then we call $\sim$ a Hopf relation.

You can guess now what the theorem is

Theorem 4. *If $\mathcal{P}$ is a hereditary family and $\sim$ is a Hopf relation on $\mathcal{P}$ then we write $H(\mathcal{P})$ for $C(\mathcal{P})$ and $H(\mathcal{P}$ is a Hopf algebra with antipode*

$$S([P]) = \sum_{k\geq 0} \sum_{\substack{x_0<\cdots<x_k \\ x_0=0_P, x_1=1_P}} (-1)^k \prod_{i=1}^{k} [x_{i-1}, x_i]$$

for $[P] \in \widetilde{\mathcal{P}}$. $H(\mathcal{P})$ is called an incidence Hopf algebra. *Furthermore, if $\mathcal{P}$ is graded then $H(\mathcal{P})$ is graded.*

Proof. $H(\mathcal{P})$ was already a coalgebra. It is an algebra as a consequence of the points above defining a Hopf relation. It is a bialgebra because if P_1 and P_2 are posets and $(x_1, x_2) \leq (y_1, y_2) \in P_1 \times P_2$ then $[(x_1, x_2), (y_1, y_2)] = [x_1, y_1] \times [x_2, y_2]$ which is what you need to calculate that the coproduct is an algebra homomorphism, and the rest of the check is easier. Finally S is the antipode because the same argument as before works. Even though $H(\mathcal{P})$ is not necessarily graded we do have $\Delta([P]) = [0_P, 0_P] \otimes P + P \otimes [1_P, 1_P] +$ stuff with all posets of size $< |P|$ which is enough to make the recursive argument work. From there Takeuchi's formula gives the S in the statement.

For more details see Schmitt. □

Finally we ended with the binomial coalgebra. Upgrading it to an algebra in this way we saw that the product tells us that $x_n = x_1^n$ so really we have polynomials in one variable, and in fact it is the same polynomial Hopf algebra as the one we discussed in lecture 4.

References

Schmitt, *Incidence Hopf Algebras*, Journal of Pure and Applied Algebra 96 (1994), 299-330, `http://home.gwu.edu/~wschmitt/papers/iha.pdf`.