# COMBINATORIAL HOPF ALGEBRAS LECTURE 7 SUMMARY

WINTER 2020

## Summary

Today we started with the graded dual.

If a vector space is graded and each graded piece is finite dimensional we say it is of *finite type*. Our examples are mostly like this but typically not finite dimensional.

**Definition 1.** *Given* $V = \bigoplus_{i\geq 0} V_i$ *with* $\dim V_i < \infty$ *let the* graded dual *of* $V$ *be*

$$V^0 = \bigoplus_{i\geq 0} V_i^*$$

This has the nice properties that the dual of finite dimensional vector spaces has, in particular the adjoint map is an isomorphism, $V^0 \otimes W^0 = (V \otimes W)^)$, adjoint maps have the same structure coefficients, graded duals of finite type algebras are coalgebras and graded duals of finite type coalgebras are algebras.

Next we discussed posets in anticipation of defining incidence coalgbras and beyond.

**Definition 2.** *A* poset *(partially ordered set) is a set with a binary relation* $\leq$ *that is reflexive* ($\forall a, a \leq a$)*, antisymmetric* ($\forall a, b, a \leq b$ *and* $b \leq q \Rightarrow a = b$)*, and transitive* ($\forall a, b, c, a \leq b$ *and* $b \leq c \Rightarrow a \leq c$)*.*

Note, we write $<$ for $\leq$ but not $=$. There are lots of examples, like $\mathbb{Z}_{\geq 1}$ with divisibility or the power set of a set with the subset relation.

**Definition 3.** *Let* $P$ *be a poset.*

- *For* $x, y \in P$ *the* interval $[x, y]$ *is the poset* $\{z \in P : x \leq z \leq y\}$ *with the same relation.*
- $P$ *is* finite *if its underlying set is finite.*
- $P$ *is* locally finite *if all of its intervals are finite.*

Check your favorite examples. How could a poset not be locally finite. Roughly, it could be very wide or very dense or very tall. We gave examples.

**Definition 4.** *Let* $P$ *be a poset and* $x, y \in P$*. We say* $x$ covers $y$ *if* $x > y$ *and there is no* $z \in P$ *with* $x > z > y$*.*

If you draw a poset going up the page with vertices for the elements and edges going up the page for the covering relation from smaller to larger, then you have a *Hasse diagram.* We gave an example.

Now we can define incidence coalgebras. First we gave the two motivating cases and then the general theorem.

Let $P$ be a locally finite poset and let $\text{Int}(P)$ be the set of intervals of $P$. Then the incidence coalgebra of $P$ is the coalgebra on $\text{Span}_K(\text{Int}(P))$ with $\Delta([x,y]) = \sum_{x\leq z\leq y}[x,z]\otimes[z,y]$ and $\epsilon([x,y]) = \begin{cases}1 & \text{if } x=y\\ 0 & \text{otherwise}\end{cases}$. You can check that this is in fact a coalgebra.

If $P$ is as above then we can form the reduced coalgebra of $P$ by in place ot $\text{Int}(P)$ taking $\text{Int}(P)/\sim$ where $\sim$ is poset isomorphism.

Now we just want to make the right definitions so that this construction works in general, as we don't need to use the set of all intervals of a fixed poset, and we can use other equivalence relations.

**Definition 5.** *A family $\mathcal{P}$ of finite intervals is* interval closed *if it is nonempty and for all $P \in \mathcal{P}$ and all $x \leq y \in P$, $[x,y] \in \mathcal{P}$.*

For $P = [x,y]$ write $0_p$ for $x$ and $1_P$ for $y$.

**Definition 6.** *An* order compatible equivalence relation *on an interval closed family $\mathcal{P}$ is an equivalence relation $\sim$ such that whenever $P \sim Q$ then there exists a bijection $f : P \to Q$ such that $[0_p, x] \sim [0_Q, f(x)]$ and $[x, 1_P] \sim [f(x), 1_Q]$.*

Two posets being order compatible doesn't necessarily imply they are isomorphic. Schmitt (see reference) has an example on p4. However usually we will use order compatible relations that are poset isomorphism along with preserving some extra structure.

The point is that these two definitions are what we need to always get a coalgebra. Specifically let $\mathcal{P}$ be an interval closed family and $\sim$ and order compatible equivalence relation. Write $\widetilde{\mathcal{P}}$ for $\mathcal{P}/\sim$ and for $P \in \mathcal{P}$ write $[P]$ for the class of $P$ (though we won't write double [s for intervals.) Let $C(\mathcal{P}) = \text{Span}_K(\widetilde{\mathcal{P}})$ with $\Delta[P] = \sum_{x\in P}[0_P, x]\otimes[x, 1_P]$ and $\epsilon([P]) = \begin{cases}1 & \text{if } |P|=1\\ 0 & \text{otherwise}\end{cases}$. Then $C(\mathcal{P})$ is a coalgebra.

As an example, a poset is a finite boolean algebra if it is isomorphic to a lattice of subsets of a finite set ordered by inclusion. Consider $\mathcal{B}$ the family of finite boolean algebras and $\sim$ poset isomorphism. Then the intervals $[U,W]$ with $U \subseteq W$ finite are determine by $|W-U|$, so let $x_{|W-U|}$ be the equivalence class. Then $C(\mathcal{B})$ is the span of the $x_i$ iwth coproduct $\Delta(x_n) = \sum_{k=0}^{n}\binom{n}{k}x_k \otimes x$. This is called the *binomial coalgebra*.

Finally, we talked about how the grading fits into the poset situation.

**Definition 7.** *A poset $P$ is* graded *if it has a rank function $r : P \to \mathbb{Z}_{\geq 0}$ such that if $x \leq y$ then $r(x) \leq r(y)$ and if $y$ covers $x$ then $r(y) = r(x)+1$.*

For posets $P$ which are finite intervals if $P$ is graded then the rank function can be shifted so $r(0_P) = 0$ still giving a rank function, and then we can define the rank of $P$ to be the rank of $1_P$. This is also the length of maximal chains from $0_P$ to $1_P$. With this shifted rank the rank is well defined on equivalence classes for order compatible equivalence relations and so if $\mathcal{P}$ is an interval closed family of graded posets with an order compatible relation then $C(\mathcal{P}) = \bigoplus_{n\geq 0} C(n)$ is a graded coalgebra where $C(n)$ is the span of the rank $n$ classes in $\widetilde{\mathcal{P}}$.

## Next time

Next class we will define and start working with incidence Hopf algebras.

## References

Schmitt, *Incidence Hopf Algebras*, Journal of Pure and Applied Algebra 96 (1994), 299-330, http://home.gwu.edu/~wschmitt/papers/iha.pdf.

Kock, *Incidence Hopf Algebras*, http://mat.uab.es/~kock/seminars/incidence-algebras.pdf

Héctor Figueroa, *Combinatorial Hopf algebras in quantum field theory I* arXiv:hep-th/0408145 section 12