COMBINATORIAL HOPF ALGEBRAS LECTURE 6 SUMMARY

WINTER 2020

Summary

Last time we'd stated and started proving

Proposition 1. (1) *Let H be a Hopf algebra. S is an antiautomorphism, that is, $S(1) = 1$ and $S(ab) = S(b)S(a)$.*
(2) *Let H be a Hopf algebra. If H is commutative or cocommutative then $S \circ S = id$.*
(3) *Let B be a graded connected bialgebra. Then B has a unique antipode S defined recursively. Furthermore S is graded and so B is a graded Hopf algebra.*

Additionally, note that the co-analogue of the first item is also true, that is $\epsilon \circ S = \epsilon$ and $\Delta(S(a)) = \sum_{(a)} S(a_2) \otimes S(a_1)$.

The idea for the proof of the first point is to work in $\mathrm{Hom}(H \otimes H, H)$ (using $\circledast$ for the convolution). Then defining $f, g, h \in \mathrm{Hom}(H \otimes H, H)$ by $f(a \otimes b) = ab$, $g(a \otimes b) = S(b)S(a)$, and $h(a \otimes b) = S(ab)$, we can compute $h \circledast f = u_H \epsilon_{H \otimes H} = f \circledast g$ and then this proves the result as $h = h \circledast (u_H \epsilon_{H\otimes H} = h \circledast f \circledast g = (u_H \epsilon_{H \otimes H}) \circledast g = g$.

For the second one check that $S \circ S$ is a right $\star$ inverse of S.

For the third one turn $S \star id = u\epsilon$ into a recursive formula, getting

$$S(a) = -a - \sum_{(a) \text{ from } \widetilde{\Delta}} S(a_1)a_2$$

for $a \in \ker \epsilon$.

Notice that applying the previous formula inductively we obtain Takeuchi's formula

$$S = \sum_{n \geq 0} (-1)^n m^{n-1} P^{\otimes n} \Delta^{n-1}$$

where P is projection onto $\ker \epsilon$ and m^{n-1}, Δ^{n-1} both mean repeated m and Δ (where order doesn't matter by associativity and coassociativity).

Next we talked about vector space duals with an emphasis on how things are very nice for the finite dimensional case but not the infinite dimensional case. Next time we will see how to salvage everything we need in the finite type case using the graded dual (both of which will be defined next time).

Definition 2. *For V a vector space the* dual *V^* is $Hom(V, K)$.*
We have the bilinear pairing $\langle \cdot, \cdot \rangle : V^* \times V \to K$ given by $\langle f, v \rangle = f(v)$.

Definition 3. *Given $f : V \to W$ K-linear, we have $f^* : W^* \to V^*$, the* adjoint map, *defined by $\langle \alpha, f(v) \rangle = \langle f^*(\alpha), v \rangle$ for $\alpha \in W^*$ and $v \in V$.*

Note that this reverses arrows so we should immediately think of applying it to algebras and coalgebras.

In the finite dimensional case if $v_1, \dots, v_n$ is a basis for V then we have the dual basis $v_1^*, \dots, v_n^*$ (sometimes notated $v^1, \dots, v^n$) defined by $v_i^*(v_j) = \delta_{i,j}$ (where δ is the Kronecker delta). In the infinite dimensional case this construction gives elements of the dual, but they are no longer a basis as their span is only the maps of finite support.

If V and W have bases $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ (not necessarily finite) and $f : V \to W$ is defined by $f(v_i) = \sum_{j\in J} a_{i,j} w_j$ then

$$\langle f^*(w_j^*), v_i\rangle = \langle w_j^*, f(v_i)\rangle = a_{i,j}$$

so if V and W are finite dimensional we get $f^*(w_j^*) = \sum_i a_{i,j} v_i^*$, but in the infinite dimensional case this doesn't tell us everything. In fact the adjoint map from $\mathrm{Hom}(V, W)$ to $\mathrm{Hom}(W^*, V^*)$ given by $f \mapsto f^*$ is an isomorphism if and only if W is finite dimensional.

Also we have the map $\rho : V^* \otimes W^* \to (V \otimes W)^*$ given by $\rho(f \otimes g)(v \otimes w) = f(v)f(w)$. This map is always injective and if V and W are finite dimensional then it is bijective.

With this we have

Proposition 4. *Let C be a coalgebra. Then C^* with $m = \Delta^* \circ \rho$ and $u = \epsilon^*$ (using the canonical identification of K with K^* by taking 1 to id) is an algebra.*

The proof is calculation. We checked one of the properties.

You would expect the dual propositive to say that if you have an algebra A you can build a coalgebra structure on A^*, but if you just try it you'll see that ρ goes the wrong way. In the finite dimensional case it works since ρ is an isomorphism (and all the proofs are even easier as you can just use the structure coefficients).

Next time

Next class we will talk about finite type vector spaces and graded duals in order to make all this work in the case we actually care about.

Then we'll start talking about posets.

References

Darij Grinberg and Victor Reiner *Hopf algebras in combinatorics*, arXiv:1409.8356.

Federico Ardila's lecture notes, file for lectures 1-9 `http://math.sfsu.edu/federico/Clase/Hopf/LectureNotes/ardilahopf1-9.pdf`