COMBINATORIAL HOPF ALGEBRAS LECTURE 5 SUMMARY

WINTER 2020

Summary

Today we started with the definition of gradings.

Definition 1. *A vector space V over K is* graded *if*

$$V = \bigoplus_{n=0}^{\infty} V_n.$$

If V and W are graded vector spaces then a map $f : V \to W$ is garded if $f(V_n) \subseteq W_n$. An algebra, coalgebra, bialgebra etc is graded if the underlying vector space and maps are all graded.

There are a few notes on this definition. This is actually the definition of $\mathbb{Z}_{\geq 0}$-grading, but we will never use other index sets for our gradings. To speak of maps that we need being graded we need to note that the tensor product of a graded vector space V is graded via $(V \otimes V)_n = \bigoplus_{i=0}^{n} V_i \otimes V_{n-i}$, and K as a vector space over itself is graded by putting it all in degree 0.

An element of a graded vector space V is *homogeneous* if it lives in some V_i.

All the examples we've seen have interesting gradings except the group ring. Both word Hopf algebras are graded by length, polynomials are graded by degree, the frist graph example is graded by number of vertices. The second is best graded by the dimension of the cycles space (aka the *loop number* in physics language, aka the first Betti number) (but also don't allow 2-valent vertices).

Definition 2. *A graded vector space V is* connected *if $V_0 = K$.*

Our grading usually comes from the combinatorics.

Definition 3. *A* combinatorial class *$\mathcal{C}$ is a set (by abuse of notation also $\mathcal{C}$) and a weight function $|\cdot| : \mathcal{C} \to \mathbb{Z}_{\geq 0}$ such that each $\mathcal{C}_n = \{c \in \mathcal{C} : |c| = n\}$ is finite.*

The Hopf algebras we play with will essentially always be defined on $\mathrm{Span}_K(\mathcal{C})$ for some combinatorial class $\mathcal{C}$. Many important classes have a unique object of size 0 (the empty object) and hence give connected Hopf algebras. Sometimes multiplication is disjoint union, then if $\mathcal{B}$ is a combinatorial class without an empty object (think of these as connected objects, but with a different sense of connected (like connected graphs)) then $K[\mathcal{B}]$ can be identified with $\mathrm{Span}_K(\mathsf{MSet}(\mathcal{C}))$ by identifying monomials with the disjoint union of their factors. here MSet is the multiset operator; it takes as input a combinatorial class and returns as output the class of all multisets of elements of the input class with weight function the sum of the weights.

This leads to the second definition of combinatorial Hopf algebra.

Definition 4 (Combinatorial Hopf algebra definition 2)**.** *A combinatorial Hopf algebra is a graded connected Hopf algebra with a distinguished basis of homogeneous elements and with nonnegative integer structure coefficients for m and Δ.*

If $\{a_i\}_{i\in I}$ is a basis then the structure coefficients are the $c_{i,j}^k$ and $d_{i,j}^k$ which define m and Δ via the following formulas

$$m(a_i, b_j) = \sum_{k\in I} c_{i,j}^k a_k$$

$$\Delta(a_k) = \sum_{i,j\in I} d_{i,j}^k (a_i \otimes a_j)$$

The point of the distinguished basis is to capture the underlying combinatorial objects. The point of requiring the structure coefficients to be nonnegative integers is to capture that the operations should be combinatorial – the structure coefficients count something.

Next we proceeded to give some useful facts.

Proposition 5. *Let B be a graded connected bialgebra over K.*

(1) *$u : K \to B_0$ is an isomorphism and $\epsilon|_{B_0}$ is the inverse.*
(2) $\ker\epsilon = \bigoplus_{n=1}^{\infty} B_n$
(3) *For $x \in \ker\epsilon$, $\Delta(x) = 1\otimes x + x\otimes 1 + \widetilde{\Delta}(x)$ where $\widetilde{\Delta}(x) \in \ker\epsilon\otimes\ker\epsilon$.*

The proof of the first part is almost just the definition of grading; one additionally can check that $\epsilon|_{B_0} \circ u = \mathrm{id}$ by the unit property of $\epsilon|_{B_0}$ as an algebra morphism. The proof of the second part is by grading and the first part, and the proof of the third part is by grading and the counit property.

Definition 6. *If $\widetilde{\Delta}(x) = 0$ then we say x is primitive. $\Delta(x) - \widetilde{\Delta}(x)$ is the primitive part of the coproduct of x.*

Proposition 7.

(1) *Let H be a Hopf algebra. S is an antiautomorphism, that is, $S(1) = 1$ and $S(ab) = S(b)S(a)$.*
(2) *Let H be a Hopf algebra. If H is commutative or cocommutative then $S \circ S = id$.*
(3) *Let B be a graded connected bialgebra. Then B has a unique antipode S defined recursively. Furthermore S is graded and so B is a graded Hopf algebra.*

We started proving this and will finish next time.

Next time

Next class we will finish the antipode property proofs and talk about duals.

References

Darij Grinberg and Victor Reiner *Hopf algebras in combinatorics*, arXiv:1409.8356.