COMBINATORIAL HOPF ALGEBRAS LECTURE 4 SUMMARY

WINTER 2020

Summary

Today we started with two things left over from last time and then did a bunch of examples. First, in order to not go crazy you want to use Sweedler notation:

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2$$

Note that the a_1 and a_2 in each term are different (at least potentially), but if we had another index to keep track of that then we'll have too much of a mess once we start doing more complicated things, so this notation, which can be a bit uncomfortable at first, is actually very convenient.

As an example, in this notation the convolution can be writen

$$(f \star g)(c) = \sum_{(c)} f(c_2)g(c_2)$$

Second, we have the following proposition

Proposition 1. *Let A be an algebra and C a coalgebra. The space of linear maps $Hom(C, A)$ is an algebra (called the* convolution algebra*) under $\star$ and $u\epsilon$.*

To prove this, we wrote out the definition of $(f\star g)\star h$ and $f\star(g\star h)$ and found associativity of both Δ_C and m_A gave the desired associativity, and then for the fact that $u\epsilon$ is the unit we wrote the commutative diagrams for the unital and counital properties of A and C on top of each other, joined by an $f \in \mathrm{Hom}(C, A)$.

Now for a bunch more examples. We already have the two word Hopf algebras from last week. Here are some others (generally I'll give these as bialgebras, as in the graded case we'll see that the antipode comes for free),

(1) $K[x]$: Product is usual multiplication of polynomials. Coproduct is $\Delta(x) = 1 \otimes x + x \otimes 1$, extended as an algebra morphism. So $\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i}$ and we also end up with $S(x) = -x$.

(2) Permutations: Let $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$, where $\mathcal{P}_n$ is the set of permutations of $\{1, 2, \ldots, n\}$, and $\mathcal{P}_0 = \{\mathbb{I}\}$. If we write the permutations in one-line notation then they are words, so we make some definitions based on the shuffle deconcatenation Hopf algebra.
   - To multiply α and β, write in one-line notation, shift all values of β up by $|\alpha|$ and then shuffle. Eg $m(12, 21) = 12 \sqcup\!\sqcup 43 = 1243 + 1423 + 1432 + 4123 + 4132 + 4312$.
   - To comultiply, write in one-line notation, deconcatenate, then *standardize*, that is keep the same total order but shift values down to consecutive values starting at 1. Eg $\Delta(1432) = \mathbb{I} \otimes 1432 + 1 \otimes 321 + 12 \otimes 21 + 132 \otimes 1 + 1432 \otimes \mathbb{I}$.

This is sometimes called the shuffle/cut Hopf algebra of permutations and is due to Malvenuto. Someone asked if we could also make a Hopf algebra of permutations based on concatention and deshuffle, and I said it would work for the same reasons. An even better argument is that it must work by duality, which we will get to soon.

(3) Group ring: this is different from the others, it is not graded and generally not combinatorial, but if you say "Hopf algebra" you should know it too. Let G be a group, then KG is the group ring, the set of finite formal sums of elements of g with multiplication the group operation extended linearly, and coproduct $\Delta(g) = g \otimes g$ for all $g \in G$ and extended linearly. The antipode is the group inverse $S(g) = g^{-1}$. This is an important part of what the intuition of an antipode is. Also, this example is why in any Hopf algebra when we have an element a with $\Delta(a) = a \otimes a$ we say a is *group like*. Because of the gradings in our key examples, we aren't going to care so much about group-like elements.

(4) A graph Hopf algebra: Let $\mathcal{G}$ be the set of unlabelled graphs. Let $H = \mathrm{Span}_K(\mathcal{G})$. H is a Hopf algebra with product given by disjoint union and coproduct given by $\Delta(G) = \sum_{S\subseteq V(G)} G[S] \otimes G[V - S]$ where $G[S]$ is the induced subgraph.

If we let $\mathcal{C}$ be the set of connected unlabelled graphs (not including an empty graph) then we can identity H as an algebra with $K[\mathcal{C}$ by identifying a monomial with the disjoint union of its parts. This is something we'll do a lot works because $\mathcal{G} = \mathsf{MSet}(\mathcal{C})$ where MSet is the multiset operator, that is it makes all objects that are multisets of objects of its argument.

(5) I wanted to do one more graph Hopf algebra, but I don't have the foundations yet to do it very cleanly. Take bridgeless graphs, again take the span of these, and again take disjoint union as the multiplication. We want the coproduct to be

$$\Delta(G) = \sum_{\substack{S \subset E(G) \\ \text{subgraph } \gamma \text{ given by} \\ S \text{ is bridgeless}}} \gamma \otimes G/\gamma$$

but there is one more caveat. In this situation we want graphs defined by their edge sets, so isolated vertices do not appear, except that all graphs with just isolated vertices can be identified with each other and give the unique empty object for this Hopf algebra. We'll return to this when we do Feynman diagrams, and then we'll have a bit of framework to make this nicer to say. The quantum field theory crowd calls this the *core Hopf algebra*.

Next time

Next class we will talk about gradings and more useful facts.

References

Federico Ardila's lecture notes, file for lectures 1-9 `http://math.sfsu.edu/federico/Clase/Hopf/LectureNotes/ardilahopf1-9.pdf`