COMBINATORIAL HOPF ALGEBRAS LECTURE 3 SUMMARY

WINTER 2020

Summary

Today we wrote down a lot of commutative diagrams and defined Hopf algebras.

For us K is a field, but in fact it can usually be a ring for things we will do.

Definition 1. *An algebra A over K is a vector space over K with two linear maps $m : A \otimes A \to A$, called the product or multiplication, and $u : K \to A$, called the unit, such that the following two diagrams commute*

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "id\otimes m"] \arrow[d, "m\otimes id"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

$$\begin{tikzcd} K \otimes A \arrow[d, "u\otimes id"] & A \arrow[l, "a\mapsto 1\otimes a"'] \arrow[r, "a\mapsto a\otimes 1"] \arrow[d, "id"] & A \otimes K \arrow[d, "id\otimes u"] \\ A \otimes A \arrow[r, "m"] & A & A \otimes A \arrow[l, "m"'] \end{tikzcd}$$

If you haven't worked with commutative diagrams before, they are digraphs with spaces labelling the vertices and maps between the appropriate spaces labelling the arcs. A directed path in the digraph is a composition of maps, and any two directed paths from the same two vertices should be equal; this is the commutativity of the diagram. Tracing this through on elements $a \otimes b \otimes c$ for instance we see the top diagram is exactly what you'd expect for associativity: top and right path gives $a(bc)$ while left and bottom path gives $(ab)c$, hence the commutativity of the diagram means $a(bc) = (ab)c$. The second diagram gives the unital property. Note that the unit in the naive sense (as an element of A) is $u(1)$.

Why do we have m as a linear map and use $\otimes$ rather than have m as a bilinear map and use $\times$? This is because the tensor product is a machine for turning linear maps into bilinear maps. In fact exactly that gives the universal property definition of the tensor product of two algebras, specifically, given vectors spaces A and B

Definition 2. *Define $A \otimes B$ as a vector space over K such that there exists a bilinear map $\iota : A \times B \to A \otimes B$ given by $\iota(a, b) = a \otimes b$, and such that for every bilinear $f : A \times B \to C$ with C also a vector space over K, we get a unique $g : A \otimes B \to C$ such that*

$$\begin{tikzcd} A \times B \arrow[rr, "\iota"] \arrow[dr, "f"] & & A \otimes B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

A statement of this form is a universal property and whenever you have one the object you are trying to describe $A \otimes B$ in this case either does not exist or it exists and is unique up to unique isomorphism. To show it exists you have to construct it, which I think you've seen in this case.

Also, we're already implicitly using that the tensor product of vector spaces is associative in order to even write down $A \otimes A \otimes A$, and we will keep using this.

To get a coalgebra we just reverse the arrows in the definition of algebra.

Definition 3. *A coalgebra C over K is a vector space over K with two linear maps $\Delta : C \to C \otimes C$, called the coproduct, and $\epsilon : C \to K$, called the counit, such that the following two diagrams commute*

$$\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "id\otimes\Delta"'] \\
C \otimes C \arrow[u, "\Delta\otimes id"'] & C \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
K \otimes C \arrow[r, "k\otimes c\mapsto kc"] & C & C \otimes K \arrow[l, "c\otimes k\mapsto kc"'] \\
C \otimes C \arrow[u, "\epsilon\otimes id"'] & C \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[u, "id"'] & C \otimes C \arrow[u, "id\otimes\epsilon"']
\end{tikzcd}$$

We can also write the notion of algebra homomorphism in this language and then get the notion of coalgebra homomorphism immediately by reversing the arrows.

Definition 4. *Let A and B be algebras over K. A linear map $f : A \to B$ is an algebra homomorphism if the following diagrams commute*

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
A \otimes A \arrow[u, "m_A"] \arrow[r, "f\otimes f"] & B \otimes B \arrow[u, "m_B"]
\end{tikzcd}$$

$$\begin{tikzcd}
& K \arrow[dl, "u_A"'] \arrow[dr, "u_B"] & \\
A \arrow[rr, "f"] & & B
\end{tikzcd}$$

Next we took a little digression into tensor products of algebras and coalgebras. I also wasn't as explicit as I should have been about what's going on with the underlying field. The thing about the field is that as with $A \otimes K$ being isomorphic to K as we used above, $K \otimes K$ is canonically isomorphic with K (the isomorphism takes $1 \otimes 1$ to 1 which is the only thing it can do), and some of the definitions below use that implictly. In fact K also has an algebra structure over itself with product that same isomorphism and with unit the identity and K has a coalgebra structure over itself with the identity as the counit and the coproduct given by $\Delta(1) = 1 \otimes 1$.

For the tensor products of algebras and coalgebras, we need the map $\tau : A \otimes B \to B \otimes A$ that swaps: $\tau(a \otimes b) = b \otimes a$.

Proposition 5. *Let A and B be algebras then $A \otimes B$ is an algebra with multiplication $m_{A\otimes B} = (m_A \otimes m_B) \circ (id \otimes \tau \otimes id)$ and unit $u_A \otimes u_B$. Similarly if C and D are coalgebras then $C \otimes D$ with coproduct $\Delta_{C\otimes D} = (id \otimes \tau \otimes id) \circ (\Delta_C \otimes \Delta_D)$ and counit $\epsilon_{C\otimes D} = \epsilon_C \otimes \epsilon_D$.*

The proof is just checking. With this in mind suppose you have a vector space B which is simultaneously an algebra and a coalgebra. Then the property that the two algebra maps are coalgebra homomorphisms is equivalent to the property that the two coalgebra maps are algebra homorphisms. The proof is just to write out the diagrams in each case and notice that they are the same, which we did. This is also a good exercise for sorting out what's going on with the field itself. Such a B is called a *bialgebra*.

Finally, then, we defined the convolution product and from there Hopf algebras.

Definition 6. *Let A be an algebra, C a coalgebra, and $f, g : C \to A$ linear maps. Then the* convolution product *of f and g, $f \star g : C \to A$ is*

$$f \star g = m \circ (f \otimes g) \circ \Delta$$

Definition 7. *A bialgebra B is a Hopf algebra if there exists a linear map $S : B \to B$, called the antipode, such that $S \star id = id \star S = u \circ \epsilon$.*

Then the first definition of a combinatorial Hopf algebra is that it is a Hopf algebra where the underlying set and all maps are combinatorially defined. This is clearly not actually a definition, and so we'll see a better definition soon.

Next time

Next class we will finish up with a few more comments on the convolution product and then lots of examples.

References

Darij Grinberg and Victor Reiner *Hopf algebras in combinatorics*, arXiv:1409.8356.