COMBINATORIAL HOPF ALGEBRAS LECTURE 2 SUMMARY

WINTER 2020

Summary

Today we talked about some Hopf algebras on words, without defining Hopf algebra.

Our set up for words is an alphabet Ω, and then Ω^* is the set of words over Ω, that is the set of finite sequences of elements of Ω including the empty word which we write 1. The length of a word w is denoted $|w|$.

We defined two products on words. The first is concatentaion

Definition 1. *Given $w_1, w_2 \in \Omega^*$, with $w_1 = a_1 a_1 \cdots a_k$ and $w_2 = b_1 b_2 \cdots b_\ell$, $a_i, b_i \in \Omega$, define the concatenation of w_1 and w_2 to be*

$$w_1 w_2 = a_1 a_2 \cdots a_k b_1 b_2 \cdots b_\ell$$

The second is shuffle. First define $W = \mathrm{Span}_K(\Omega^*)$ where K is a field.[1] Then we have

Definition 2. *Given $w_1, w_2 \in \Omega^*$, with $w_1 = a_1 a_1 \cdots a_k$ and $w_2 = a_{k+1} a_{k+2} \cdots a_{k+\ell}$, $a_i \in \Omega$, define the shuffle of w_1 and w_2 to be*

$$w_1 \sqcup\!\sqcup\, w_2 = \sum_{\substack{\sigma \in S_{k+\ell} \\ \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(k) \\ \sigma^{-1}(k+1) < \sigma^{-1}(k+2) < \cdots < \sigma^{-1}(k+\ell)}} a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(k+\ell)}$$

and extend linearly to W.

We can also extend concatenation linearly to W.

We wrote down some examples and also noted that the shuffle can be described recursively $aw_1 \sqcup\!\sqcup\, bw_2 = a(w_1 \sqcup\!\sqcup\, bw_2) + b(aw_1 \sqcup\!\sqcup\, w_2)$ and $w \sqcup\!\sqcup\, 1 = w = 1 \sqcup\!\sqcup\, w$ for all $a, b \in \Omega$ and $w_1, w_2, w \in \Omega^*$.

Some things to note about these products. The unit in both cases is 1. Shuffle is commutative and concatenation is not. W has a preferred basis given by Ω^* and concatenation in Ω^* stays in Ω^* while the same is not true for shuffle. Also both are graded. We'll come back to all of these.

Next we looked at some coproducts on words. It is often useful to think in terms of decomposing objects rather than in terms of building them. One way to capture this algebraically is with a coproduct. These two coproducts are dual to the products defined above

[1] Though mostly we can work with K a ring, and $\mathbb{Z}$ is a nice example with combinatorial use. If you are in to that kind of thing, keep an eye out for anywhere that I actually use that K is a field as opposed to a ring.

Definition 3. *Given $w \in \Omega^*$, $w = a_1 a_2 \cdots a_k$ define the deconcatenation coproduct of w to be*

$$\Delta_{dc}(w) = \sum_{i=0}^{k} a_1 a_2 \cdots a_i \otimes a_{i+1} a_{i+2} \cdots a_k$$

and extend linearly to W.

Definition 4. *Given $w \in \Omega^*$, $w = a_1 a_2 \cdots a_k$ define the deshuffle coproduct of w to be*

$$\Delta_{dc}(w) = \sum_{S \subseteq \{1,\ldots,k\}} a_S \otimes a_{\{1,\ldots,k\}-S}$$

where for $T \subseteq \{1, \ldots, k\}$, $T = \{i_1 < i_2 < \ldots, i_\ell\}$, $a_T = a_{i_1} a_{i_2} \cdots a_{i_\ell}$, and extend linearly to W.

Note that for both coproducts $\Delta(1) = 1 \otimes 1$ and $\Delta(a) = 1 \otimes a + a \otimes 1$ for any $a \in \Omega$. As we'll discuss further next week, both are also graded and deshuffle is cocommutative but deconcatenation is not.

Next we talked about which pairs of product and coproduct are compatible. By compatible we want the coproduct to be an algebra morphism, as we'll define formally next time. You checked that concatenation with deconcatentation is not compatible and neither is shuffle with deshuffle. Two single letter words are sufficient for a counterexample. The other two pairs are compatible.

The last thing we talked about was the map S

Definition 5. *Define $S : W \to W$ by $S(a_1 a_2 \cdots a_k) = (-1)^k a_k a_{k-1} \cdots a_1$ for $a_i \in \Omega$ and extend linearly.*

This map has the property that for both comaptible pairs of product and coproduct (letting m denote the product)

- $m(\mathrm{id} \otimes S)\Delta(w) = 0$ for $w \in \Omega^*, w \neq 1$.
- $m(\mathrm{id} \otimes S)\Delta(1) = 1$.

You can prove this inductively just by calculating. I didn't present the proof particularly well because we got a little pressed for time, so you might want to try it yourself, though what I wrote down is correct.

Next time

Next class we will actually define Hopf algebras. There will be many commutative diagrams.

References

You could find these examples in many places, see for instance p19 and p22 of my book "A combinatorial perspective on quantum field theory".