COMBINATORIAL HOPF ALGEBRAS LECTURE 19 SUMMARY

WINTER 2020

Summary

We did our first online class today. The quality was much worse than in our test last week, so we'll try different software next time. I'll make the summary more complete on account of the change of medium.

Δ and S in other bases. First we discussed Δ in other bases, you got into groups and worked out

$$\begin{aligned}
\Delta(e_n) &= \sum_{k=0}^{n} e_k \otimes e_{n-k} \quad \text{using } e_n = m_{\underbrace{(1,1,1,\dots,1)}_{n \text{ times}}} \\
\Delta(p_n) &= p_n \otimes 1 + 1 \otimes p_n \quad \text{using } p_n = m_{(n)} \\
\Delta(h_n) &= \sum_{k=0}^{n} h_{n-k} \otimes h_k
\end{aligned}$$

The last one is hardest to see from the monomial symmetric functions and later in the class we'll see a nicer way to get it.

We know Sym has an antipode since it is graded and connected, and we have the usual recursive formula

$$S(f) = -f - \sum_{\substack{(f) \\ \text{nonprimitive part}}} S(f_1)f_2$$

From this we can compute directly

$$S(p_n) = -p_n$$

The others take a little more work so we made a proposition

Proposition 1.

$$S(e_n) = (-1)^n h_n$$

Proof. Apply the usual recursive formula

$$\begin{aligned}
S(e_n) &= -e_n - \sum_{i=1}^{n-1} S(e_i)e_{n-i} \\
&= -e_n - \sum_{i=1}^{n-1} (-1)^i h_i e_{n-i} \quad \text{inductively} \\
&= -e_n + e_n + (-1)^n h_n \quad \text{by claim below} \\
&= (-1)^n h_n
\end{aligned}$$

where we needed the claim which some of you may know

$$\text{Claim: } \sum_{i=0}^{n} (-1)^i h_i e_{n-i} = \begin{cases} 0 & n > 0 \\ 1 & n = 0 \end{cases}$$

The proof of the claim is by generating series. Let $H(t) = \sum_{n\geq 0} h_n t^n$ and $E(t) = \sum_{n\geq 0} e_n t^n$ then by usual generating function stuff

$$H(t) = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t} \quad \text{and} \quad E(t) = \prod_{i=1}^{\infty} (1 + x_i t)$$

and so $H(-t)E(t) = 1$. Extracting coefficients gives the claim. □

As a corollary of the proposition we get

$$S(h_n) = (-1)^n e_n$$

since $S^2 = \text{id}$ in the commutative case, and we get a quicker way to calculate

$$\begin{aligned} \Delta(h_n) &= \Delta(S((-1)^n e_n)) \\ &= (-1)^n \sum_{i=0}^{n} S(e_{n-i} \otimes S(e_i) \\ &= \sum_{i=0}^{n} h_{n-i} \otimes h_i \end{aligned}$$

Sym as an incidence Hopf algebra. Let $\mathcal{L}$ be the set of finite linear orders and let $\sim$ be poset isomorphism.

This is not yet hereditary so let $\mathcal{L}^*$ be finite products of elements of $\mathcal{L}$.

Then we have a hereditary family. Poset isomorphism always automatically has the properties we need, so we have a Hopf relation. Hence we have an incidence Hopf algebra.

Up to isomorphism $\mathcal{L}$ has one element of each size, call the one of size n, ℓ_n. As posets these are the ladder trees, that is rootes trees where all vertices have at most 1 child.

The product is poset product. This is the free commutative product on the ℓ_i.

The coproduct is $\Delta(\ell_n) = \sum_{i=0}^{n} \ell_i \otimes \ell_{n-i}$ because the coproduct in an incidence Hopf algebra sums over intermediate z and then takes initial and final segments.

So, there are two nice isomorphisms with Sym, $\ell_i \mapsto e_i$ or $\ell_i \mapsto h_i$.

Self-duality. Recall the graded dual: Given $V = \bigoplus_{i=0}^{\infty} V_i$ with each V_i finite dimensional, we have the graded dual $V^\circ = \bigoplus_{i=0}^{\infty} V_i^*$.

There is a natural inner product on Sym given by

$$\langle m_\lambda, h_\mu \rangle = \begin{cases} 1 & \lambda = \mu \\ 0 & \lambda \neq \mu \end{cases}$$

Then we view h_μ as the characteristic function of m_μ, and hence we identify Sym_n^* with Sym_n.

Now Sym° is (by things we showed before) also a Hopf algebra, the thing is to show that it is Sym itself.

Proposition 2. *$Sym^\circ = Sym$ using the identification from the previous slide. Therefore Sym is self dual.*

Proof. It suffices to check that the structure coefficients match on a basis. Let's use the m_λ on the Sym side and h_λ on the Sym° side. Then

$$\begin{aligned}\langle \Delta(m_\lambda), h_\mu \otimes h_\nu\rangle &= \langle \sum_{\alpha \dot\cup \beta = \lambda} m_\alpha \otimes m_\beta, h_\mu \otimes h_\nu\rangle \\ &= \begin{cases} 1 & \lambda = \mu \dot\cup \nu \\ 0 & \text{otherwise} \end{cases} \\ &= \langle m_\lambda, h_{\mu\dot\cup\nu}\rangle \\ &= \langle m_\lambda, h_\mu h_\nu\rangle \end{aligned}$$

as desired. □

QSym. Consider again a function of finite degree

$$f(x) = \sum c_{\underline{\eta}} \underline{x}^{\underline{\eta}} \in K[[x_1, x_2, \ldots]]$$

If $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ and $x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}$ have the same coefficient whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots j_k$, then we say f is quasisymmetric.

Eg $x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \cdots = \sum_{i<j} x_i^2 x_j$ is quasisymmetric but not symmetric.

Let QSym be the set of quasisymmetric functions.

For $\alpha = (\alpha_1, \ldots, \alpha_k)$ a composition, define the *monomial quasisymmetric function* to be

$$M_\alpha = \sum_{i_1 < \cdots i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$$

This is quasisymmetric almost directly from the definition.

Note that symmetric functions are quasisymmetric so $\mathrm{Sym} \subseteq \mathrm{QSym}$.

QSym is a vector space because addition also preserves the quasisymmetry property.

The set $\{M_\alpha\}_{\alpha \text{ composition}}$ gives a vector space basis of QSym so $\dim \mathrm{QSym}_n = 2^{n-1}$.

QSym as an algebra. We checked this but it didn't write very well in Bongo. What is the point? Given a monominal m in fg for $f, g \in \mathrm{QSym}$, for each way to write m as a product of $m_1 m_2$ with m_1 a monomial in f and m_2 a monomial in g, then thinkng of the monomials as ordered (x_1 before x_2 etc), then m is a particular shuffle of m_1 and m_2. Any monomial m' in fg with the same composition of exponents can be split into m_1' and m_2' according to the same shuffle giving m_1' with the same composition of exponents as m_1 and similarly for m_2'. The quasisymmetry of f means the coefficients of m_1 and m_1' are the same and similarly for the m_2. The same holds for every way of obtaining m. Thus the coefficients of m and m' are the same and so fg is quasisymmetric.

QSym is a coalgebra with the $\Delta(f(\underline{x})) = f(\underline{y}, \underline{z})$ coproduct. It is still a composition of algebra maps so QSym is a bialgebra in the same way as Sym. QSym is graded by degree and is connected so QSym is a Hopf algebra. The inclusion $\mathrm{Sym} \subseteq \mathrm{QSym}$ is a Hopf algebra map.

References

Two places to find a presentation of symmetric functions that focuses on the Hopf side of things is Federico Ardila's Hopf algebra course lectures 25 to the end http://tinyurl.com/ardilahopf, and the book by Grinberg and Reiner, "Hopf Algebras in Combinatorics", arXiv:1409.8356