COMBINATORIAL HOPF ALGEBRAS LECTURE 18 SUMMARY

WINTER 2020

Summary

Today we started the third part of the course with a whirwind review of symmetric functions.

Symmetric functions start with $K[[x_1, x_2, \ldots]]$. Let S_∞ be the group of permutations on $\mathbb{Z}_{\geq 1}$ which leave all but finitely many numbers fixed. Then S_∞ acts on $K[[x_1, x_2, \ldots]]$ by permuting the variables and the *ring of symmetric functions*, Sym, is the set of finite degree elements of $K[[x_1, x_2, \ldots]]$ which are invariant under the S_∞ action. It is a ring under the usual operations on formal power series.

Sym is graded by degree: $\mathrm{Sym} = \bigoplus_{n\geq 0} \mathrm{Sym}_n$ where Sym_n is the set of symmetric functions which are homogeneous of degre n.

There are many important bases for Sym. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n.

- Monomial symmetric functions: $m_\lambda = \sum_{\underline{\alpha}\in S_\infty(\lambda_1,\ldots,\lambda_k,0,0,\ldots)} \underline{x}^{\underline{\alpha}}$ where we're using multiindex notation. $\{m_\lambda\}_{\lambda \text{ a partition of} n}$ is a basis for Sym_n.
- Elementary symmetric functions: $e_n = e_{\underbrace{(1, 1, \ldots, 1)}_{\text{n times}}}$ and $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$. Then it turns out that $\{e_\lambda\}_{\lambda \text{ a partition of } n}$ is a basis for Sym_n, and $\mathrm{Sym} = K[e_1, e_2, \ldots]$ that is the e_n are free commutative algebra generators for Sym.
- Homogeneous symmetric functions: $h_n = \sum_{\lambda \text{ a partition of } n} m_\lambda$, $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$, again this is a basis and the h_n are free commutative algebra generators.
- Power sum symmetric functions: $p_n = m_{(n)}$ and p_λ defined likewise and with the same properties (in this case requiring that K has characteristic 0.)
- Schur functions:

$$s_\lambda = \sum_{\substack{T \text{ semistandard} \\ \text{filling of shape } \lambda}} \prod_{i\geq 1} x^{\text{number of } i\text{s in } T}$$

  where the shape (aka Ferrers diagram or Young diagram) is made by putting λ_1 boxes in a row, then λ_2 and so on, and a semistandard filling is a way of putting positive itegers in the boxes so that they are weakly increasing along the rows and strictly down the columns. It isn't obvious this is a symmetric function, but it is and gives another basis.

Next we talked about the coproduct. The idea is $\Delta(f(x)) = f(y, z)$. To do this, first take any bijection from $\{x_1, x_2, \ldots\}$ to $\{y_1, y_2, \ldots, z_1, z_2, \ldots\}$ (by the symmetry it won't matter which bijection we take when applied to symmetric functions). This gives a map $\mathrm{Sym}(x_1, x_2, \ldots) \to \mathrm{Sym}(y_1, y_2, \ldots, z_1, z_2, \ldots)$. Next we have a maps $\mathrm{Sym}(y_1, y_2, \ldots, z_1, z_2, \ldots) \to \mathrm{Sym}(y_1, y_2, \ldots) \otimes \mathrm{Sym}(z_1, z_2, \ldots)$ given by in each monomial simply putting the y part on the left of the tensor and putting the z part onthe right. Composing these two maps gives our coproduct.

We noticed that $\Delta(m_\lambda) = \sum_{\mu \cup \nu = \lambda} m_\mu \otimes m_\nu$ where the sum is over all (ordered) partitions of the parts of λ into μ and ν. We checked that Δ has the required properties so that we have a graded connected bialgebra and hence a Hopf algebra.

Next time we'll discuss the coproduct on other bases and the antipode. This will be online because of the coronavirus closure.

References

This is all standard stuff. Two places to find a presentation of symmetric functions that focuses on the Hopf side of things is Federico Ardila's Hopf algebra course lectures 25 to the end `http://tinyurl.com/ardilahopf`, and the book by Grinbery and Reiner, "Hopf Algebras in Combinatorics", arXiv:1409.8356