COMBINATORIAL HOPF ALGEBRAS LECTURE 17 SUMMARY

WINTER 2020

Summary

We started with a few more examples of scalar Feynman integrals. Then we went over the sheet about integrating the banana (which you can find on the course website).

Then we talked a bit about Feynman integrals beyond the massless scalar case. They get more complicated, but the basic points remain in how they are built up out of the graphs, but the contribution of each edge gets a lot more complicated. The superficial degree of divergence still represents the leading behaviour of the edge's contribution.

We finished off with discussing more of the mathematical context. The classical Riemann-Hilbert problem begins with a simple closed curve Σ in the complex plane, which divides the plane into the inside Σ_+ and the outside Σ_-. The problem is to find functions M_+ and M_- analytic on Σ_+ and Σ_- respectively and which appropriately glue on Σ, one way to set up "appropriately glue" is to suppose α, β, c are given and to require $\alpha(z)M_+(z) + \beta(z)M_-(z) = c(z)$.

This has been generalized in lots of directions, you can take Σ not simple, you can take other manifolds, you can consider the matrix valued case. In the matrix valued case it becomes a matrix factorization problem and is called Birkhoff factorization.

This last direction can also be extended to the bialgebra case, which is what we want

Definition 1. *Let B be a bialgebra and let $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ be an algebra with a decomposition into two vector spaces. Then given $F : B \to \mathcal{A}$ with $F(1) = 1$, a Birkhoff decomposition of F is a pair $F_+, F_- : B \to \mathcal{A}$ with $F_+(1) = F_-(1) = 1$ such that $F = F_-^{\star -1} \star F_+$ and $F_\pm(\ker \epsilon) \subseteq \mathcal{A}_\pm$.*

Proposition 2. *Let B be a connected bialgebra and $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ and F as above. Then F admits a unique Birkhoff decomposition which can be recursively computed by $F_-(x) = -R(\bar{F}(x))$ and $F_+(x) = (id - R)(\bar{F}(x))$ for $x \in \ker \epsilon$ where $F : \mathcal{A} \to \mathcal{A}_-$ is projection and $\bar{F}(x) = F(x) + \sum_{\substack{(x) \\ \text{non prim part}}} F_-(x_1)F(x_2)$.*

In physics language this $\bar{F}$ is the Bogoliubov R-bar operator.

The proof of the proposition is first note that if F_- and F_+ satisfies the convolution part of the definition of a Birkhoff decomposition then it is recursively defined by the formulae of the proposition. Then it only remains to check that $F_\pm(\ker \epsilon) \subseteq \mathcal{A}_\pm$. For F_- this is immediate since $F_-(\ker \epsilon)$ is in the image of F, and then for F_+ just use $F_+(x) = (\mathrm{id} - R)(\bar{F}(x))$.

In renormalization if F is the Feynman rules then F_- is the counterterms as we defined them before (i.e. S_R^F) since $R(\bar{F})$ gives exactly the antipode-like recursive definition of S_R^F. Then F_+ is the renormalized value as by definition $F_+ = F_- \star F$. The idea is if $\mathcal{A}_-$ is all the bad stuff (eg divergent stuff) then F_+ will only take good values, that is it will be renormalized.

There is one last detail. Today R was projection but in the tree toy model R was evaluation at $s = 1$. How to square these two things? In fact we can define a generalized Birkhoff decomposition for other linear maps R, with $\mathrm{im}(R)$ in place of $\mathcal{A}_-$ and $\ker(R)$ in place of $\mathcal{A}_+$. The property R needs to satisfy is

$$R(x)R(y) = R(R(x)y) + R(xR(y)) + \lambda R(xy).$$

A linear map with this property is called a Rota-Baxter operator of weight λ. Then with this R the same formulas as in the proposition define the generalized Birkhoff decomposition of F, and this is what we usually want for renormalization.

Two examples: if we set $D = 4 - 2\epsilon$ then our Feynman rules take values in a space of Laurent series in ϵ (this is *dimensional regularization*) and we use $R(\sum_{i=-L}^{\infty} c_i\epsilon^i) = \sum_{i=-L}^{-1} c_i\epsilon^i$ (this is *minimal subtraction*). My favorite example is renormalization by subtraction like we saw in the toy model. Then F takes values in some space of formal integrals and R is evaluation at a fixed reference point.

References

The Birkhoff stuff comes from the initial Connes-Kreimer collaboration. You can find a nice exposition in Erik Panzer's masters thesis arXiv:1202.3552, section 2.2