COMBINATORIAL HOPF ALGEBRAS LECTURE 16 SUMMARY

WINTER 2020

Summary

From last time we had

Definition 1. *Let G be a graph in a combinatorial physical theory. The* superficial degree of divergence *of G is*

$$D\ell - \sum_{e \text{ internal}} w(e) - \sum_{v} w(v)$$

where ℓ is the loop order of G. If the superficial degree of divergences id nonnegative we say the graph is divergent*.*

We say a combinatorial physical theory is *renormalizable* if the superficial degree of divergence only depends on the multiset of external edges and only finitely many external edge multisets are divergent.

Now divergent multisets of external edges are multisets of half-edge types, and vertex and edge types are also multisets of half-edge types. In the vertex case this works nicely, if all the divergent multisets of external edges of size ≥ 3 are also vertex types then if we contract a divergent subgraph with at least 3 external edges then we are left with a vertex in the theory.

So if our theory is renormalizable but doesn't contain these divergent multisets as vertices we should add these to our theory (with power counting weight set to be the superficial degree of divergence).

The edge situation is slightly more subtle. We don't have 2-valent vertices in our graphs currently. The idea is we want to contract the subgraph but then replace the resulting two edges in series with a 2-valent vertex in the middle by a single edge. To make this make sense with the half edge types, note that so far our edge types have been of two forms, two identical half edges giving an unoriented edge type, and a front and back half giving an oriented edge type. In particular no half edge type is in more than one edge type, so given a half edge we can uniquely say what the "other half" would be. Then to contract a subgraph with two external edges, we want to excise the entire subgraph and join the other halves of the two external edges into one new edge. This is possible within the theory if the multiset formed of the other halves of the divergent multiset we started with is an edge type of the theory. If these are not all in the theory, add them as discussed above.

There is a physical caveat. In QED, the superficial degree of divergence calculation says that three external photons (and no other external edges) would also be divergent, but we don't add this as a vertex. This is because there is a symmetry that makes the Feynman integrals for graphs with three external photons vanish. This is called Furry's theorem.

Now we can define the renormalization Hopf algebra for a renormalizable combinatorial physical theory (with extra edge and vertex types added as above if necessary) as follows.

Let $\mathcal{G}$ be the set of divergent 1PI Feynman diagrams in the theory. Begin with $K[\mathcal{G}]$ with the coproduct

$$\Delta(G) = \sum_{\substack{\gamma \subseteq G \\ \gamma \text{ product of 1PI} \\ \text{divergent subgr.}}} \gamma \otimes G/\gamma$$

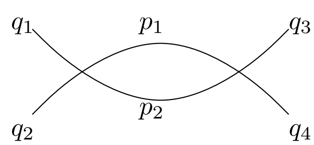
and the usual counit, and graded by loop order, and taking the quotient by setting all graphs consisting of a single vertex with external edges to 1. The result is graded and connected so is a Hopf algebra.

Next we looked at how to write massless scalar Feynman integrals. There were two ways to do it (equivalent by a little thought about flows in graphs).

Fix a graph whose masseless scalar Feynman integral we want to write down.

- Choose an arbitrary orientation for the edges.
- Associate to each internal edge a variable (which we will think of as taking values in $\mathbb{R}^4$); call the variable for edge e, p_e
- Associate to each external edge a variable; call the variable for edge e, q_e
- Make the integrand by multiplying by $\frac{1}{p_e^2}$ for each edge e where in the usual physics way, p_e^2 means $p_e \cdot p_e$, and multiplying by a Dirac delta function for each vertex v, $\delta(\sum_{e\sim v} \pm p_e)$, where $e \sim v$ means e incident to v, and the sign is $+$ if the edge is directed into v and $-$ otherwise.
- Integrate over all p_e.

For example, if we have



with all edges directed from left to right across the page, then the Feynman integral is

$$\int d^4p_1 d^4p_2 \frac{\delta(q_1 + q_2 - p_1 - p_2)\delta(p_1 + p_2 - q_3 - q_4)}{p_1^2 p_2^2}$$

However, we can use up the delta functions to get rid of the p_2 integration: the second delta function says $p_2 = q_3 + q_4 - p_1$ and then with this substitution the first delta function says $q_1 + q_2 = q_3 + q_4$, so the integral becomes

$$\delta(q_1 + q_2 - q_3 - q_4) \int d^4p_1 \frac{1}{p_1^2(q_3 + q_4 - p_1)^2}$$

The outer delta is saying that we have overall momentum conservation. (Each delta says we have momentum conservation at that delta's vertex).

We could have written it this way from the start, specifically another way to write the massless scalar Feynman integral is as follows.

- Choose an arbitrary orientation for the edges.
- Associate to each external edge a variable; call the variable for edge e, q_e
- Choose a basis of oriented cycles for the cycle space and associate a variable k_i to each C_i in the basis.

- Choose a routing of the external momenta through the graph (specifically, choose a tree joining the external edges, then with overall momentum conservation, the amount of external momentum running through each edge of the tree is the signed sum of the external momenta on one side of the cut induced by that edge.)
- For each edge e let p_e be the signed sum of the k_i for cycles of the basis involving e and the q_i as described in the previous point if e has external momenta routed through it.
- Integrate over the k_i.

References

How to write Feynman integrals is the sort of thing you will find in a quantum field theory textbook. For example, Itzykson and Zuber "Quantum field theory", seee p265 in the Dover edition.