COMBINATORIAL HOPF ALGEBRAS LECTURE 15 SUMMARY

WINTER 2020

Summary

We gave some more examples of combinatorial physical theories which you can find in the first reference below. Then we talked about secret labelled counting, specifically if $\mathcal{L}$ is some class of labelled graphs (in the sense of this section of the course) and f is the map forgetting the labelling, and $\mathcal{G} = f(\mathcal{L})$ then by elementary group theory

$$\sum_{G \in \mathcal{L}} \frac{x^{|G|}}{|G|!} = \sum_{G \in \mathcal{G}} \frac{x^{|G|}}{|\mathrm{Aut}(G)|}$$

where here $|G|$ is the number of half edges of G. The automorphism group is the group of automorphisms of the set of half edges which preserve the graph structure. Using the physicists conventions that the external edges are always labelled, we are interested in automorphisms which fix each external edge.

The same identity of generating series holds if we have any weight function ϕ (say Feynman rules) which depends only on $f(G)$

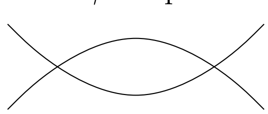
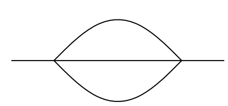
$$\sum_{G \in \mathcal{L}} \frac{\phi(G) x^{|G|}}{|G|!} = \sum_{G \in \mathcal{G}} \frac{\phi(G) x^{|G|}}{|\mathrm{Aut}(G)|}$$

Generally any time you see generating series with $1/|\mathrm{Aut}|$ then you have secret labelled counting and typical labelled things like going from connected to not necessarily connected by exponentiation all hold.

For this section, the correct notion of subgraph γ of a graph G is that

- γ is a graph whose half edge set is a subset of the half edge set of G,
- γ is full on the vertices in the sense that if a half edge of a vertex of G is in γ then all the half edges of that vertex are in γ and that vertex is a vertex of γ, and all vertices of γ arise in this way.
- All internal edges of γ are internal edges of G.

Note that the converse of the second item is not true even when restricted to edges both of whose half edges are in γ. In particular

 is a subgraph of 

in three different ways.

Contracting a subgraph means contracting its internal edges.

A graph is 1PI (one particle irreducible) if it is 2-edge connected, that is removing any internal edge does not disconnect it. Often we care about graphs whose connected components are all 1PI, that is bridgeless graphs.

Definition 1. *The* core *Hopf algebra (what the second Hopf algebra from the second lecture was meant to be) is defined as follows. Take $K[\mathcal{C}]$ where $\mathcal{C}$ is the set of nonempty 1PI graphs (in the sense of this section). The coproduct is*

$$\Delta(G) = \sum_{\substack{\gamma\subseteq G \\ \gamma \text{ bridgeless}}} \gamma \otimes G/\gamma$$

and extended as an algebra homomorphism. The counit is as usual. Now consider all the graphs that consist of a single vertex with some number of external edges. Mod out by the ideal formed by these graphs. The result is a Hopf algebra graded by the loop order (ie the dimension of the cycle space) called the core Hopf algebras.

Next we looked at divergence in Feynman graphs. We will write our Feynman integrals in momentum space, so in particular the integration variables will be momenta (in space-time, so energy-momentum really). Ultraviolet (UV) divergences are divergences when momenta are large. Infrared (IR) divergences are divergences when variables are small.

As variables get large we can consider just the degree of their leading monomials. Collecting these together gives the superficial degree of divergence.

Definition 2. *Let G be a graph in a combinatorial physical theory. The* superficial degree of divergence *of G is*

$$D\ell - \sum_{e \text{ internal}} w(e) - \sum_{v} w(v)$$

where ℓ is the loop order of G. If the superficial degree of divergences id nonnegative we say the graph is divergent*.*

We say a combinatorial physical theory is *renormalizable* if the superficial degree of divergence only depends on the multiset of external edges and only finitely many external edge multisets are divergent.

We looked at some QED examples. QED is renormalizable.

References

My PhD thesis, arXiv:0810.2249, section 2.2

Kreimer, "The core Hopf algebra", arXiv:0902.1223.