COMBINATORIAL HOPF ALGEBRAS LECTURE 14 SUMMARY

WINTER 2020

Summary

We did some computations in the toy model that we introduced last time, namely, if $\mathcal{H}$ is the Connes-Kreimer Hopf algebra, then we define

$$F_s(B_+(f)) = \int_0^\infty \frac{F_z(f)}{s+z}dz$$

where s is a parameter with $s > 0$ and $F_s(1) = 1$, $F_s(t_1 \cdots t_k) = F_s(t_1)\cdots F_s(t_k)$. The maps F_s is the *Feynman rules* for this toy model.

Now, note that $F_s(\bullet) = log(s+z)|_0^\infty = \infty$. Let's think of this as a problem of reference point; relative values may make sense even when absolute ones do not. So compare the integrands of $F_s(\bullet)$ and $F_1(\bullet)$

$$\frac{1}{s+z} - \frac{1}{1+z} = \frac{1-s}{(s+z)(1+z)}$$

and this we can integrate $z = 0 \ldots \infty$

$$\int_0^\infty \frac{1-s}{(s+z)(1+z)}dz = -\log(s)$$

(we did the actual integrals in maple, and you are welcome to do so too, though if you're up on your calculus these ones aren't hard.)

We will write $F_s(\bullet) - F_1(\bullet) = -\log(s)$, but what we mean with the subtraction on the left is subtract the integrands and then integrate.

Next we tried $F_s(B_+(\bullet))$. This time $F_s(B_+(\bullet)) - F_1(B_+(\bullet))$ does not converge. The problem is that just the integral for the lower vertex already diverges. We need to take care of this inner integral too, and we will do this recursively by a subtraction as well.

Let R be the map given by evaluation of s at 1. Then define

$$S_R^{F_s}(t) = -R(F_s(t)) - \sum_{\substack{C \text{ antichain} \\ \emptyset \neq C \neq \{\text{root}\}}} S_R^F(\prod_{v\in C} t_v) R(F_s(t - \prod_{v\in C} t_v))$$

for a tree t and extend to $\mathcal{H}$ as an algebra homomorphism. We call $S_R^{F_s}$ the *counterterm* for t.

Then the *renormalized Feynman rules* are

$$F_{\text{ren}} = S_R^{F_s} \star F_s.$$

We tried this out on $B_+(\bullet)$ using maple. It is satisfying to see it actually give a finite answer ($\log^2(s)/2$ in this case). Note an important detail in all these cases, when you choose the integration variable for each vertex you need to make the same choice in all the integrands, so you should consider the tree as decorated by the integration variables and carry that decoration through the calculation of $S_R^{F_s}$ and F_{ren}.

The next step is to do the same thing for actual Feynman integrals, but first we need Feynman graphs.

For the rest of this section of the course a graph is

- a set H of half edges
- a partition of H into parts of size ≥ 3 giving vertices
- a partition of H into parts of size ≤ 2 giving internal edges (parts of size 2) and external edges (parts of size 1).

This is reminiscent of combinatorial maps. A *combinatorial physical theory* is

- a set of half edge types
- a set of edge types consisting of unordered pairs of half edge types defining the allowable edges in the theory
- a set of vertex types consisting of multisets of half edge types defining the allowable vertices in the theory
- an integer *power counting weight*, w, associate dot each edge and vertex type
- a dimension D of space-time.

Then a Feynman graph in a combinatorial physical theory is a graph in the sense above along with a map from H to the set of half edge types so that every internal edge and every vertex are among the allowable edge and vertex types.

There's a subtety regarding labelling. So far this is set up with an underlying set H, hence it is about labelled graphs (labelled on the half-edges). Physicists usually don't label the half edges except that they do label the external edges. We'll talk more about labelling next time.

We gave just one example: in quantum electrodynamics (QED) there are three half edge types, the half-photon, the front half-fermion, and the back half-fermion. There are two edge types, the photon, made of a pair of half-photons, and the fermion, made of a front half-fermion and a back half-fermion. There is one vertex consisting of one of each half-edge types. The power counting weights are 2 for the photon, 1 for the fermion and 0 for the vertex. The dimension of space time is 4.

References

Erik Panzer's masters thesis arXiv:1202.3552, section 3.1.

This stuff about combinatorial physical theories is how I formulate it, so you're stuck reading something I wrote for a reference. My PhD thesis will do, arxiv.org:0810.2249, section 2.2