COMBINATORIAL HOPF ALGEBRAS LECTURE 12 SUMMARY

WINTER 2020

Summary

Today we discussed B_+ and Hochschild 1-cocycles.

B_+ is the add-a-root operator. $B_+(t_1 \cdots t_k)$ is the tree with a new root vertex and the subtrees at the root are t_1, t_2, $\ldots$, t_k. We observed that

$$\Delta B_+ = (\mathrm{id} \otimes B_+)\Delta + B_+ \otimes 1$$

This means that B_+ is a Hochschild 1-cocycle.

Actually we defined a family of Hochschild cohomologies based on a left and a right coaction. This required some definitions.

Definition 1.
- *Let C be a coalgebra over K. Then a left C-comodule M is a vector space over K with a map $\psi_L \in Hom(M, C \otimes M)$ such that $(id \otimes \psi_L) \circ \psi_L = (\Delta \otimes id) \circ \psi_L$ and $(\epsilon \otimes id) \circ \psi_L = id$.*
- *A right comodule is defined similarly with a map ψ_R and left and right sides of tensors switched in the above definition.*
- *A bicomodule is both a left and a right comodule with $(id \otimes \psi_R) \circ \psi_L = (\psi_L \otimes id) \circ \psi_R$.*

Now we make a Hochschild cochain complex as follows

Definition 2. *Let C be a coalgebra and M a C-bimodule. The Hochschild cochain complex $(HC^\bullet(M), \delta^\bullet)$ is for $k \in \mathbb{Z}_{\geq 0}$*
- *$HC^k(M) = Hom(M, C^{\otimes k})$, these are the spaces of cochains*
- *$\delta^k : HC^k(M) \to HC^{k+1}(M)$, these are the coboundary maps, defined by*

$$\begin{aligned}\delta^k(L) =& (id \otimes L) \circ \psi_L \\ &+ \sum_{i=1}^{k} (-1)^i (id^{\otimes i-1} \otimes \Delta \otimes id^{\otimes k-i}) \circ L \\ &+ (-1)^{k+1} (L \otimes id) \circ \psi_R\end{aligned}$$

Then we sketched the check that this is in fact a cochain complex, that is, that $\delta^{n+1} \circ \delta^n = 0$ (in usual cohomology fashion, this is written $\delta^2 = 0$, we just don't write the indices when it is clear from context). I slightly messed up talking through the check. The point is to consider each term in the definition of $\delta^k(L)$ composed with each term in δ^{k+1}, and note that the results each appear twice with opposite signs. I was missing at that moment what happens when both terms are the first or the last. The answer is that by the first of the left or right comodule properties these match with one coproduct term. You can find all the details in section 2.4 of the reference.

So the cohomology groups are $H^k(HC^\bullet(M), \delta^\bullet) = \ker \delta^k / \mathrm{im}\delta^{k-1}$, and you want to use these to understand what you started with. Actually we'll only use the very first non-trivial piece, namely the 1-cocycles. L is a 1-cocycle if $L \in HC^1(M)$ and $\delta(L) = 0$.

It remains to consider which maps ψ_L and ψ_R to use. We'll now move to the context of a bialgebra B or Hopf algebra as a bicomodule over itself. We can use Δ for either ψ_L or ψ_R or both, but another possiblity is $\phi_L = 1 \otimes \mathrm{id}$ or $\psi_R = \mathrm{id} \otimes 1$ and all four combinations of these possible maps do in fact give a bicomodule structure of B over itself.

We made a little table of what the 1-cocycles look like in all four possibilities, if both ψ_L and ψ_R are Δ you get a coderivation. If $\psi_l = \Delta$ and $\psi_R = \mathrm{id} \otimes 1$ then you get the property that B_+ has, and if $\psi_L = 1 \otimes \mathrm{id}$ and $\psi_R = \mathrm{id} \otimes 1$ then you get an endomorphism with image in primitive elements.

That was interesting but actually an aside because from now on we'll restrict to $\psi_l = \Delta$ and $\psi_R = \mathrm{id} \otimes 1$, and B_+ will be our prototypical 1-cocycle. In fact there is an important universality result.

Theorem 3. *Let $\mathcal{H}$ be the Connes-Kreimer Hopf algebra and let A be a commutative algebra and $L : A \to A$ a map. Then there exists a unique algebra homomorphism $\rho_L : \mathcal{H} \to A$ such that $\rho_L \circ B_+ = L \circ \rho_L$. If further A is a bialgebra and L is a Hochschild 1-cocycle then ρ_L is a bialgebra homomorphism. If even further A is a Hopf algebra then ρ_L is a Hopf algebra homomorphism.*

We'll use this theorem in two ways, both important. First when A is a Hopf algebra then this is saying that for pairs of a commutative Hopf algebra and a 1-cocycle the Connes-Kreimer Hopf algebra with B_+ is universal. Second, when A is the target algebra for Feynman rules then this theorem is explaining that if we know what B_+ should do after applying Feynman rules then we know the Feynman rules themselves (here ρ_L would be the Feynman rules).

References

Erik Panzer's masters thesis section 2.4 arXiv:1202.3552.