COMBINATORIAL HOPF ALGEBRAS LECTURE 11 SUMMARY

WINTER 2020

Summary

Today we discussed the Connes-Kreimer Hopf algebra, and linked to the previous topic by also seeing it as an incidence Hopf algebra.

An *antichain* A in a poset P is a set such that there is no pair $a \neq b \in A$ with $a \leq b$ or $b \leq a$.

Let $\mathcal{T}$ be the set of rooted trees with no plane structure and no empty tree. We talked about a few different ways of defining rooted trees, since we've been so poset-y lately, for this summary let's say a rooted tree is a finite poset where there is a unique maximal element, the root, and all other elements have exactly one element covering them.

Then let $\mathcal{H} = \mathbb{Q}[\mathcal{T}]$ as usual identifying multiplication with disjoint union so we can see this as an algebra of forests. Note that the empty tree/forest has returned as the empty monomial 1. This is the algebra structure of the Connes-Kreimer Hopf algebra. Next for the coalgebra structure, for $t \in \mathcal{T}$ define

$$\Delta(t) = \sum_{\substack{C \subseteq V(t) \\ C \text{ antichain}}} \left(\prod_{v \in C} t_v \right) \otimes \left(t \smallsetminus \prod_{v \in C} t_v \right)$$

and extended as an algebra homorphism, where t_v is the subtree rooted at v. Note that you should interpret $t - \prod_{v \in C} t_v$ as 1 if $\prod_{v \in C} t_v = t$. The counit is, as usual, $\epsilon(1) = 1$ and $\epsilon(t) = 0$ for $t \in \mathcal{T}$. This is graded by number of vertices and it connected so the antipode follows as usual. This is the *Connes-Kreimer Hopf algebra*.

We talked about two other ways to write down the coproduct, the one you most usually see is in terms of admissible cuts which are the edges immediately rootwards of the $v \in C$ the way I have defined it. Then you have one little notational oddity that you still need to have a cut when v is just the root, so you just call this the full cut. You can also write the coproduct in terms of cuts of total orders refining the partial order of the tree.

We did some examples and you did an example of Δ and S.

Then we looked at how to view $\mathcal{H}$ as an incidence Hopf algebra. Given a tree t, it is a poset, but it isn't the right poset. The poset we want is the poset of "things which can be cut off of t" ordered by inclusion. This is a special case of a general construction in order theory, but we need a few definitions to get there.

An *order ideal* of a poset P is a subset I of P such that if $x \in I$ and $y \leq x$ then $y \in I$.

The "things which can be cut off" are the order ideals of the tree.

A *lattice* L is a poset where each pair of elements s, t has a meet $s \wedge t \in L$ (or greatest lower bound) and a join $s \vee t \in L$ (or least upper bound), where greatest lower bound and least upper bound are defined in the obvious way, i.e. $s \wedge t \leq s$, $s \wedge t \leq t$ and if $x \leq s, x \leq t$ then $x \leq s \wedge t$ and analogously for $\vee$.

A lattice L is *distributive* if for all $s, t, u \in L$, $s \wedge (t \vee u) = (s \wedge t) \vee (s \wedge u)$ and the same with $\wedge$ and $\vee$ swapped.

Given a finite poset P let J_P be the poset of order ideals of P ordered by inclusion. Then it turns out that J_P is a distrubutive lattice and (a result of Birkhoff) every finite distributive lattice is the J_P for some poset P.

Now returning to the Connes-Kreimer Hopf algebra, given a tree t, take J_t and take the incidence Hopf algebra on J_t, then this has the same coproduct as t, and so the Connes-Kreimer Hopf algebra is the incidence algebra of the intervals in all J_t and their products.

References

That way of writing the Connes-Kreimer coproduct is the way you'll find it in things I write, eg in "A combinatorial perspective on quantum field theory"

For the incidence connection see Section 13 of Hector Figueroa, Jose M. Gracia-Bondia, *Combinatorial Hopf algebras in quantum field theory I*, arXiv:hep-th/0408145