COMBINATORIAL HOPF ALGEBRAS LECTURE 10 SUMMARY

WINTER 2020

Summary

Today we talked about Möbius inversion and antipodes.

First we set up the classical situation.

Definition 1. *The classical Möbius function is*

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ has a repeated prime factor} \\ (-1)^r & \text{if } n \text{ has } r \text{ distinct prime factors} \end{cases}$$

Then classical Möbius inversion is

Proposition 2. *If* $f(n) = \sum_{d|n} g(d)$ *then* $g(n) = \sum_{d|n} f(d)\mu(n/d)$.

A nice classical example is Euler's *phi*-function $\phi(n)$ defined to be the number of integers between 1 and n which are relatively prime with n. Then $n = \sum_{d|n} \phi(n)$ so by Möbius inversion $\phi(n) = \sum_{d|n} d\mu(n/d)$.

The next piece of the classical picture is $\zeta : \mathbb{Z}_{\geq 1} \to K$ defined by $\zeta(n) = 1$. The Dirichlet generating series of ζ is the Riemann zeta function. In general the Dirichlet generating series of a sequence $\alpha : \mathbb{Z}_{\geq 1} \to K$ is $A(s) = \sum_{n \geq 1} \alpha(n)/n^s$.

If we have two Dirichlet series $F(s) = \sum_{n\geq 1} f(n)/n^s$ and $G(s) = \sum_{n \geq 1} g(n)/n^s$ then note that

$$F(s)G(s) = \sum_{n \geq 1} \left(\sum_{i \cdot j = n} f(i)g(j) \right) \frac{1}{n^s}$$

So if we write $f \star g)(n) = \sum_{i \cdot j = n} f(i)g(j)$ then we have a convolution product that gives the coefficient of the Dirichlet series of the product.

With this notation Möbius inversion becomes

$$f = g \star \zeta \quad \Rightarrow \quad g = f \star \mu$$

and we see that μ is the convolution inverse of ζ (where the identity of $\star$ is δ defined by $\delta(1) = 1$ and $\delta(n) = 0$ for $n > 1$).

This gives a fun way to calculate that the Dirichlet series of ϕ is $\zeta(s-1)/\zeta(s)$.

Next we looked at the general picture. Let $\mathcal{P}$ be a hereditary family with a Hopf morphism. Consider the convolution algebra $\mathrm{Hom}(H(\mathcal{P}, K)$, i.e. the incidence algebra.

Define $\zeta([x, y]) = 1$ for all $[x, y] \in \mathcal{P}$ and extend to $\mathrm{Span}(\widetilde{\mathcal{P}})$. Define μ to be the convolution inverse of ζ. Then we get

Proposition 3.

$$\mu = \zeta \circ S$$

so μ essentially is the antipode, and in particular a nice formula for the antipode gives a nice formula for μ. The proof of the proposition is a short calculatation.

Then Möbius inversion itself is as before

Proposition 4. *Given $f, g \in Hom(H(\mathcal{P}), K)$ we have*

$$f = g \star \zeta \quad \Rightarrow \quad g = f \star \mu$$

The proof is a quick calculation.

Finally we looked at some nested examples. When P is a poset $\mathcal{P}$ is the set of intervals of P and $\sim$ is poset isomorphism, then we get the usual poset version of Möbius inversion like you might find in CO630 (depending on who taught it and how much time they had). One subexample is the classic Möbius inversion with $P = (\mathbb{Z}_{\geq 1}, |)$. The binomial Hopf algebra is another nice subexample as there the antipode is $S(x_n) = (-1)^n x_n$ so $\mu(U, V) = \zeta(S(x_{|U-V|})) = (-1)^{|U-V|}$. A subsubexample is Inclusion-Exclusion.

References

Federico Ardila's lecture 20 `http://math.sfsu.edu/federico/Clase/Hopf/LectureNotes/ardilahopf20.pdf`

Joachim Kock, Incidence Hopf algebras `http://mat.uab.es/~kock/seminars/incidence-algebras.pdf`