INTEGRATING THE BANANA

CO739, WINTER 2020

Let's try integrating the double edge graph. We'll use arbitrary dimension D and arbitrary powers r and s of the propagators. Also we need an analytic identity (Feynman parameters):

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(ax+b(1-x))^{\alpha+\beta}} \qquad \text{for } \alpha, \beta > 0$$

where Γ is the gamma function, the unique smooth interpolation of the factorial with $\Gamma(n) = (n-1)!$. One other way to define Γ is $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ for $\mathrm{Re}(z) > 0$, and then extended by analytic continuation. Γ has poles at $0, -1, -2, \ldots$, and is otherwise nonzero. Another useful identity is $\Gamma(x)\Gamma(-x) = \frac{-\pi}{x\sin(\pi x)}$.

Now calculate

$$\begin{aligned}
&\int d^Dk \frac{1}{(k^2)^r((k+q)^2)^s} \\
&= \int d^Dk \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 dx \frac{x^{r-1}(1-x)^{s-1}}{(xk^2+(1-x)(k+q)^2)^{r+s}} \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 dx x^{r-1}(1-x)^{s-1} \int d^Dk \frac{1}{\big(xk^2+(1-x)(k+q)^2\big)^{r+s}} \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 dx x^{r-1}(1-x)^{s-1} \int d^Dk \frac{1}{\big((k+q(1-x))^2+q^2(x-x^2)\big)^{r+s}} \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 dx x^{r-1}(1-x)^{s-1} \int d^Dk \frac{1}{(k^2+q^2(x-x^2))^{r+s}} \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 dx x^{r-1}(1-x)^{s-1} \int_0^\infty d|k| \frac{|k|^{D-1}}{(|k|^2+q^2(x-x^2))^{r+s}} \int d\Omega_k \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \int_0^1 dx x^{r-1}(1-x)^{s-1} \int_0^\infty d|k| \frac{|k|^{D-1}}{(|k|^2+q^2(x-x^2))^{r+s}} \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \frac{\Gamma(r+s-\frac{D}{2})\Gamma(\frac{D}{2})}{2\Gamma(r+s)} (q^2)^{\frac{D}{2}-r-s} \int_0^1 dx x^{\frac{D}{2}-1-s}(1-x)^{\frac{D}{2}-1-r} \\
&= \frac{\pi^{\frac{D}{2}}\Gamma(r+s-\frac{D}{2})}{\Gamma(r)\Gamma(s)} (q^2)^{\frac{D}{2}-r-s} \frac{\Gamma(\frac{D}{2}-r)\Gamma(\frac{D}{2}-s)}{\Gamma(D-r-s)}
\end{aligned}$$

when $2r+2s > D > 0$, $D > 2r > 0$, and $D > 2s > 0$, and where by usual physics convention the square of an element in $\mathbb{R}^D$ means its dot product with itself, and $d\Omega_k$ refers to the angular integration over the unit $D-1$-sphere in $\mathbb{R}^D$.

So we see that taking $D = 4-2\epsilon$ and taking $r = s = 1+\epsilon$ are not so different, but the integral diverges for $D = 4$, $r = s = 1$.