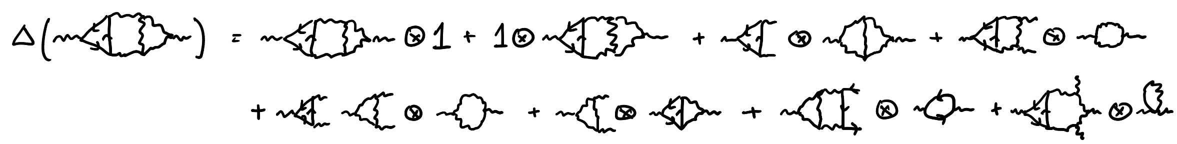
COMBINATORIAL HOPF ALGEBRAS, WINTER 2020, ASSIGNMENT 3

SOLUTIONS

(1) (a)



(b) We discussed two maps R, evaluation of an integral at a fixed value (take the specific case of the toy model on trees for concreteness) and more briefly we discussed *minimal subtraction* where $R(\sum_{i=-L}^{\infty} c_i\epsilon^i) = \sum_{i=-L}^{-1} c_i\epsilon^i$. Both these maps R are Rota-Baxter maps. Check this and determine the weight of the Rota-Baxter map in each case.

(2) The Rota Baxter property is

$$R(x)R(y) = R(R(x)y) + R(xR(y)) + \lambda R(xy).$$

Consider the map R which evaluates an integrand with s as a free parameter at $s = 1$ (and we write this with the outer integrals as usual, even though we have to do those integrals at the end).

Then

$$\begin{aligned} &R\left(\int_0^\infty dz \frac{F(z)}{s+z}\right) R\left(\int_0^\infty dw \frac{G(w)}{s+w}\right) \\ &= \left(\int_0^\infty dz \frac{F(z)}{1+z}\right)\left(\int_0^\infty dw \frac{G(w)}{1+w}\right) \end{aligned}$$

and

$$\begin{aligned} &R\left(R\left(\int_0^\infty dz \frac{F(z)}{s+z}\right)\int_0^\infty dw \frac{G(w)}{s+w}\right) + R\left(\int_0^\infty dz \frac{F(z)}{s+z} R\left(\int_0^\infty dw \frac{G(w)}{s+w}\right)\right) \\ &= R\left(\left(\int_0^\infty dz \frac{F(z)}{1+z}\right)\int_0^\infty dw \frac{G(w)}{s+w}\right) + R\left(\int_0^\infty dz \frac{F(z)}{s+z}\left(\int_0^\infty dw \frac{G(w)}{1+w}\right)\right) \\ &= \left(\left(\int_0^\infty dz \frac{F(z)}{1+z}\right)\int_0^\infty dw \frac{G(w)}{1+w}\right) + \left(\int_0^\infty dz \frac{F(z)}{1+z}\left(\int_0^\infty dw \frac{G(w)}{1+w}\right)\right) \end{aligned}$$

because the one integrand is a constant with respect to the variable of the other one

$$= 2\left(\int_0^\infty dz \frac{F(z)}{1+z}\right)\left(\int_0^\infty dw \frac{G(w)}{1+w}\right).$$

Additionally

$$R\left(\int_0^\infty dz\frac{F(z)}{s+z}\int_0^\infty dw\frac{G(w)}{s+w}\right)$$
$$=\left(\int_0^\infty dz\frac{F(z)}{1+z}\int_0^\infty dw\frac{G(w)}{1+w}\right)$$

So with $\lambda = -1$ we have that R is a Rota Baxter operator.

The second example is similar but easier. Let $R(\sum_{i=-\lambda}^{\infty} c_i x^i) = \sum_{i=-\lambda}^{-1} c_i x^i$ and let $x = \sum_{i=-\lambda}^{\infty} c_i x^i$ and $y = \sum_{i=-\mu}^{\infty} d_i x^i$. Then

$$R(x)R(y) = (\sum_{i=-\lambda}^{-1} c_i x^i)(\sum_{i=-\mu}^{-1} d_i x^i)$$

and

$$\begin{aligned}
&R(R(x)y) + R(xR(y))\\
&= R((\sum_{i=-\lambda}^{-1} c_i x^i)(\sum_{i=-\mu}^{\infty} d_i x^i)) + R((\sum_{i=-\lambda}^{\infty} c_i x^i)(\sum_{i=-\mu}^{-1} d_i x^i))\\
&= 2(\sum_{i=-\lambda}^{-1} c_i x^i)(\sum_{i=-\mu}^{-1} d_i x^i) + R((\sum_{i=-\lambda}^{-1} c_i x^i)(\sum_{i=0}^{\infty} d_i x^i)) + R((\sum_{i=0}^{\infty} c_i x^i)(\sum_{i=-\mu}^{-1} d_i x^i)).
\end{aligned}$$

Additionally

$$R(xy) = (\sum_{i=-\lambda}^{-1} c_i x^i)(\sum_{i=-\mu}^{-1} d_i x^i) + R((\sum_{i=-\lambda}^{-1} c_i x^i)(\sum_{i=0}^{\infty} d_i x^i)) + R((\sum_{i=0}^{\infty} c_i x^i)(\sum_{i=-\mu}^{-1} d_i x^i))$$

since $R((\sum_{i=0}^{\infty} c_i x^i)(\sum_{i=0}^{\infty} d_i x^i)) = 0$.

This gives that R is Rota Baxter with $\lambda = -1$.

(3) As an example, consider a scalar field theory with a k valent vertex of weight 0 and edges of weight 2. Let G be a graph in the scalar field theory with a k valent vertex. Let G have e internal edges, q external edges, and loop number ℓ. For the theory to be renormalizable we need $D\ell - 2e$ to depend only on the number of external edges. By Eulers formula along with regularity we have

$$e(2-k) + k\ell = k - q$$

so for the theory to be renormalizable we need $D = 2k/(k-2)$. If $k = 4$ then the theory is renormalizable in $D = 4$ while if $k = 3$ then the theory is renormalizable in $D = 6$, and if $k = 6$ then the theory is renormalizable in $D = 3$. No other values of $k > 2$ give a theory with an integer dimension of spacetime since the only possible common factors of k and $k-2$ are 2 and 1, so with the 2 already present in the numerator, the largest possible denominator giving an integer answer is 4.

Did anyone notice this is wrong (because I forgot $k = 6$) in my booklet "A combinatorial perspective on quantum field theory"?

(4) The idea here is that we should map the x variables to the y and z variables so that all the y variables come before all the z variables. Then when filling a shape or a

skew shape with the y and z variables, all the ys will appear up and to the left of all the z variables.

Note that we can think of a filling as a filling by the variables rather than by the indices of the variables, and we will use this language or perspective in the below.

Say we take the shape μ/ν. Fix a filling in the y and z variables. Add all the boxes with y variables to ν, call this shape α. The claim is that α is also a partition shape. If not, then some box of α has a box not in alpha to its left or above it. This box must still be in μ, but can't be in ν since ν is in α. Therefore this box must have been filled with a z variable. But this contradicts that all y variables come before all z variables. Hence α is a partition shape.

This construction gives that the y variables give a filling of α/ν and the z variables give a filling of μ/α. Additionally take any filling of α/ν with y variables and any filling with μ/α with z variables. This gives a filling of μ/nu since all y variables come before all z variables, and the same holds for any parition shape α which is contained in μ and contains ν.

From this we almost get that

$$\Delta(\mu/\nu) = \sum_{\nu \leq \alpha \leq \mu} \alpha/\nu \otimes \mu/\alpha,$$

which is what we want.

The remaining lacuna is as follows. We defined fillings (and hence Schur functions) in terms of the order structure of the positive integers, but we used filling defined in terms of the order structure of a concatenation of two copies of the positive integers. It remains to show that this was legitimate. Note that we can't just appeal to the action of the symmetric group since no finite permutation can bring the one order structure to the other.

One way to resolve this problem is to note that for any fixed shape if we restrict to a finite number of x variables which is at least twice the number of boxes then we have at least one monomial of each type appearing in that Schur function. Since the monomial symmetric functions are a basis for symmetric functions, knowing what the coproduct looks like on these terms determines the coproduct for the entire Schur function. Now consider a bijection from all xs to ys and zs such that the finite number of x variables restricted to above is mapped to a list of y variables followed by a list of z variables, in each case at least as many variables as boxes. We can pick such a bijection such that both domain and codomain have the same order structure, and when restricting as above this gives the coproduct described above with the appropriate restrictions on y and z. Since there are at least as many ys as boxes and as many zs as boxes, all decompositions of μ/ν into μ/α and α/ν are represented.

Now for the antipode you can use the recursive formula or Takeuchi with the coproduct as described above. The better answer is that $S(s_\lambda) = (-1)^{|\lambda|} s_{\lambda^t}$ where ^{t} indicates to transpose the shape. I hope you played around enough to notice this formula, but the proof is beyond this course.