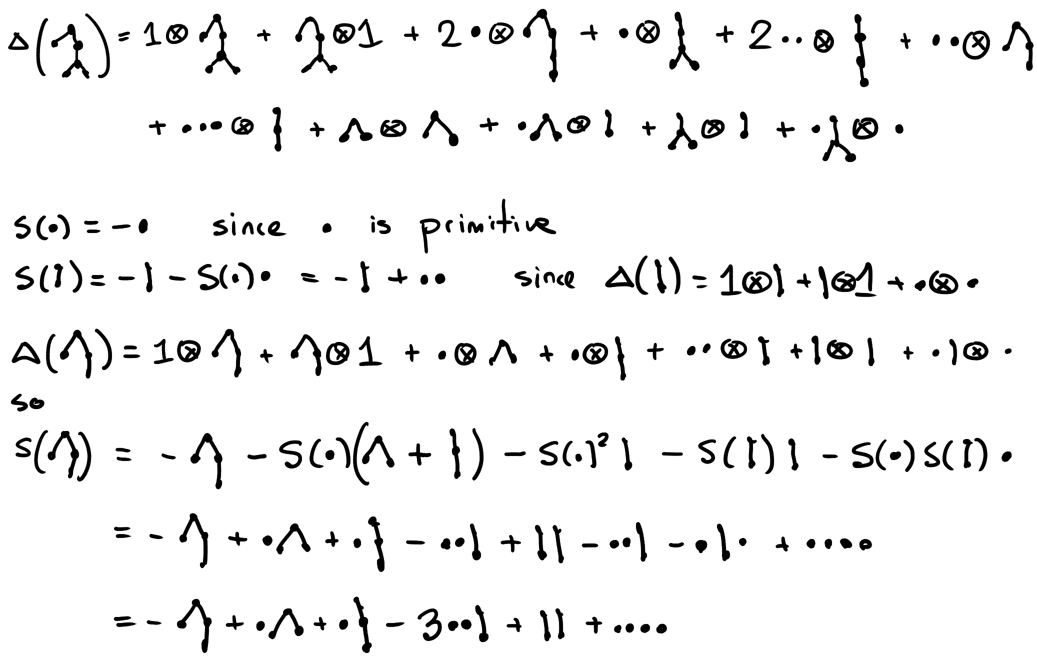
COMBINATORIAL HOPF ALGEBRAS, WINTER 2020, ASSIGNMENT 1

DUE THURSDAY MARCH 5 IN CLASS

(1) (a) I was too lazy to type this, so here is a hand version



(b) First let's calculate the bits and pieces that we need. Let the whole tree be t. I will label the root by a and the two leaves by b and c. The integration variable for a will always be called z and the integration variables for the leaves will be called z_1 and z_2 respectively.

$$S_R^{F_s}(b\bullet) = -RF_z(b\bullet) = -\int_0^\infty \frac{dz_1}{s+z_1}$$

$$S_R^{F_s}(c\bullet) = -\int_0^\infty \frac{dz_2}{s+z_2}$$

$$\begin{aligned}
S_R^{F_s}(t) &= -RF_s(t) - S_R^{F_s}(b\bullet)RF_s(B_+(c\bullet)) - S_R^{F_s}(c\bullet)RF_s(B_+(b\bullet)) - S_R^{F_s}(c\bullet b\bullet)RF_s(a\bullet) \\
&= -\int_0^\infty\int_0^\infty\int_0^\infty \frac{dz dz_1 dz_2}{(1+z)(z+z_1)(z+z_2)} + \int_0^\infty\int_0^\infty\int_0^\infty \frac{dz dz_1 dz_2}{(1+z)(1+z_1)(z+z_2)} \\
&\quad + \int_0^\infty\int_0^\infty\int_0^\infty \frac{dz dz_1 dz_2}{(1+z)(z+z_1)(1+z_2)} - \int_0^\infty\int_0^\infty\int_0^\infty \frac{dz dz_1 dz_2}{(1+z)(1+z_1)(1+z_2)}
\end{aligned}$$

$$\begin{aligned}
F_{\text{ren}}(t) &= (S_R^{F_s} \star F_s)(t) \\
&= S_R^{F_s}(t) + F_s(t) + S_R^{F_s}(b\bullet)F_s(B_+(c\bullet)) + S_R^{F_s}(c\bullet)F_s(B_+(b\bullet)) + S_R^{F_s}(b\bullet c\bullet)F_s(a\bullet) \\
&= \int_0^\infty\int_0^\infty\int_0^\infty dz dz_1 dz_2 \Bigg(-\frac{1}{(1+z_1)(z+z_1)(z+z_2)} + \frac{1}{(1+z)(1+z_1)(z+z_2)} \\
&\quad + \frac{1}{(1+z)(z+z_1)(1+z_2)} - \frac{1}{(1+z)(1+z_1)(1+z_2)} + \frac{1}{(s+z)(z+z_1)(z+z_2)} \\
&\quad - \frac{1}{(s+z)(1+z_1)(z+z_2)} - \frac{1}{(s+z)(z+z_1)(1+z_2)} + \frac{1}{(s+z)(1+z_1)(1+z_2)} \Bigg)
\end{aligned}$$

Now let's give all this to Maple

```
> assume(s>0);
> assume(z>0);
> assume(z1>0);
> assume(z2>0);
> integrand := - 1/((1+z)*(z+z1)*(z+z2))\
>  + 1/((1+z)*(1+z1)*(z+z2)) + 1/((1+z)*\
> (z+z1)*(1+z2)) + 1/((s+z)*(z+z1)*(z+z2\
> )) - 1/((s+z)*(1+z1)*(z+z2)) - 1/((s+z\
> )*(z+z1)*(1+z2)) - 1/((1+z)*(1+z1)*(1+z2)) + 1/((s+z)*(1+z1)*(1+z2))\;
                                  1                                    1
integrand := - ------------------------------ + ------------------------------
               (1 + z~) (z~ + z1~) (z~ + z2~)   (1 + z~) (1 + z1~) (z~ + z2~)

                      1                                  1
     + ----------------------------- + -------------------------------
       (1 + z~) (z~ + z1~) (1 + z2~)   (s~ + z~) (z~ + z1~) (z~ + z2~)

                      1                                  1
     - ------------------------------ - ------------------------------
       (s~ + z~) (1 + z1~) (z~ + z2~)   (s~ + z~) (z~ + z1~) (1 + z2~)

                     1                                 1
     - ---------------------------- + ----------------------------
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```
      (1 + z~) (1 + z1~) (1 + z2~)   (s~ + z~) (1 + z1~) (1 + z2~)

> factor(normal(integrand));
                                           2
                                  (z~ - 1)  (s~ - 1)
     - --------------------------------------------------------------
        (1 + z~) (z~ + z1~) (z~ + z2~) (1 + z1~) (1 + z2~) (s~ + z~)
> int(factor(normal(integrand)), z2=0..infinity);
                                     2               2      2   2          3
- ln(z~) (s~ z~ - s~ - z~ + 1)/(s~ z1~  z~ + s~ z1~ z~  + z1~  z~  + z1~ z~

           2                       2      2               2     3
    + s~ z1~  + 2 s~ z1~ z~ + s~ z~  + z1~  z~ + 2 z1~ z~  + z~  + s~ z1~

                       2
    + s~ z~ + z1~ z~ + z~ )

> factor(int(factor(normal(integrand)), z2=0..infinity));
                             ln(z~) (z~ - 1) (s~ - 1)
                   - -----------------------------------------
                     (z~ + z1~) (1 + z~) (1 + z1~) (s~ + z~)

> int(factor(int(factor(normal(integrand)), z2=0..infinity)), z1=0..infinity);
                                         2
                                   ln(z~)  (s~ - 1)
                                - ---------------------
                                               2
                                  s~ z~ + z~  + s~ + z~

> factor(int(factor(int(factor(normal(in\
> tegrand)), z2=0..infinity)), z1=0..infinity));
                                         2
                                   ln(z~)  (s~ - 1)
                                 - ------------------
                                   (1 + z~) (s~ + z~)

> int(factor(int(factor(int(factor(norma\
> l(integrand)), z2=0..infinity)), z1=0..infinity)), z=0..infinity);
                                         2     2
                              -1/3 (ln(s~)  + Pi ) ln(s~)
```

(2) (a) Let $A = K[l_1, l_2, \ldots]$ and $\mathcal{H}$ be the Connes-Kreimer Hopf algebra. A is a sub algebra of $\mathcal{H}$ by definition. Next note that $\Delta(l_n) = \sum_{i=0}^{n} l_i \otimes l_{n-i}$, so A is closed under the coproduct. It is clearly also closed under the counit and by

the recursive formula then it is also closed under S. Therefore A is a sub Hopf algebra.

(b) There are many ways to do this. Here's a direct way with a little counting. Let $p_n = [x^n]\log\left(\sum_{i=0}^{\infty} l_i x^i\right)$

$$\begin{aligned}[x^n]\log\left(\sum_{i=0}^{\infty} l_i x^i\right) &= [x^n]\sum_{i\geq 1}\frac{(-1)^i \sum j \geq 1 l_j x^j)^i}{i} \\ &= \sum_{i\geq 1}\frac{(-1)^i}{i}\sum_{n_1+n_2+\cdots+n_i=n} l_{n_1}l_{n_2}\cdots l_{n_i}\end{aligned}$$

So

$$\Delta(p_n) = \sum_{i\geq 1}\frac{(-1)^i}{i}\sum_{n_1+n_2+\cdots+n_i=n}\sum_{\substack{0\leq j_1\leq n_1\\ 0\leq j_2\leq n_2\\ \vdots \\ 0\leq j_i\leq n_i}} l_{j_1}l_{j_2}\cdots l_{j_i}\otimes l_{n_1-j_1}l_{n_2-j_2}\cdots l_{n_i-j_i}$$

Now consider some $l_{j_1}l_{j_2}\cdots l_{j_s}\otimes l_{k_1}l_{k_2}\cdots l_{k_t}$. We want to count how many times this appears. Suppose there are s trees on the left and t on the right. The arguments which follow are symmetric so we may suppose that $s \leq t$.

If $s = 0$ then the only way this term can appear is from (half of) the primitive part of $\Delta(l_{k_1}l_{k_2}\cdots l_{k_s})$, and so this must appear with the same coefficient as it does in p_n itself.

If $s = 1$, then there are two ways the term can appear. Either l_{j_1} is grafted below one of the trees of the right hand side (grafted above the left in the reverse argument when $t = 1$), or the term comes from the coproduct of a forest of the form $l_{j_1}l_{k_1}\cdots l_{k_t}$. There are t ways the first case can occur and each of them appears with coefficient $(-1)^t/t$. There are $t+1$ ways the second case can occur and each of them appears with coefficient $(-1)^{t+1}/(t+1)$. Together the coefficient is $t(-1)^t/t + (t+1)(-1)^{t+1}/(t+1) = 0$.

Something similar holds in general. Consider $1 \leq s \leq t$. There are $t(t-1)\cdots(t-s+1) = t!/(t-s)!$ ways to graft the left hand trees under the right hand trees. There are $s\cdot(t+1)\cdot t!/(t-s+1)! = (t+1)!/(t-s+1)!$ ways to put one of the left hand trees among the right hand trees and graft the rest, and so on. The overall coefficient is (where the index m is the number of left hand trees put among the right hand trees and the rest are grafted under).

$$\begin{aligned}
&\sum_{m=0}^{s}\binom{s}{m}\frac{(-1)^{t+m}}{t+m}(t+m)(t+m-1)\cdots(t+1)\frac{t!}{(t-s+m)!}\\
&=\sum_{m=0}^{s}\binom{s}{m}(-1)^{t+m}\frac{(t+m-1)!}{(t-s+m)!}\\
&=-\frac{d^{s-1}}{dz}\sum_{m=0}^{s}\binom{s}{m}(-z)^{t+m-1}|_{z=0}\\
&=-\frac{d^{s-1}}{dz}(-z)^{t-1}\sum_{m=0}^{s}\binom{s}{m}(-z)^{m}|_{z=0}\\
&=-\frac{d^{s-1}}{dz}(-z)^{t-1}(1-z)^{s}|_{z=0}\\
&=0
\end{aligned}$$

since $s \geq 1$ and an $1-z$ remains after the $s-1$ derivatives.

This calculation tells us that all the non-primitives terms in the coproduct cancel out, while the calculation for $s=0$ (and the analogous calculation for $t=0$) give that the primitive part of the coproduct is correct and so each p_n is primitive.

(3) Let L be a 1-cocycle in the Hopf algebra of polynomials in x. Then by the 1-cocycle property $\Delta(L(1)) = (\mathrm{id}\otimes L)\Delta(1) + L(1)\otimes 1 = 1\otimes L(1) + L(1)\otimes 1$ so $L(1)$ is primitive. A polynomial of degree >1 cannot be primitive because the leading term will contribute non-primitive terms to the coproduct and these cannot be cancelled by other terms by gradedness. So all primitive must have the form $ax+b$, but then calculating explicitly we see that $b=0$, so $L(1)=ax$ for some $a\in K$.

Now I claim by induction that $L(x^n) = \frac{a}{n+1}x^{n+1}+$ lower order terms. The base case holds by the calculation above. Consider $L(x^k)$. From the cocycle property we have

$$\Delta(L(x^k)) = \sum_{i=0}^{k}\binom{k}{i}x^i\otimes L(x^{k-i}) + L(x^k)\otimes 1$$

By the inductive hypothesis and gradedness we see that $L(x^k)$ is a polynomial of degree $k+1$. Suppose the leading term of $L(x^k)$ is cx^{k+1}. Then

$$\Delta(L(x^k)) = \sum_{j=0}^{k+1}c\binom{k+1}{j}x^j\otimes x^{k+1-j} + \text{lower order terms}$$

So by the inductive hypothesis we have $c\binom{k+1}{i} = \binom{k}{i}\frac{a}{k+1-i}$ for each $0\leq i\leq k$. Each of these gives $c=\frac{a}{k+1}$, which proves the claim.

Note that this first claim tells us that L behaves like integration with perhaps something lower order as well. It remains to sort out the lower order stuff.

My next claim is that for $2\leq i\leq n$ we have

$$[x^i]L(x^n) = \frac{n(n-1)\cdots(n-i+2)}{i!}[x]L(x^{n-i+1})$$

Note that this implies that $[x^n]L(x^n) = [x]L(x)$, so the subleading term always has the same coefficient.

This is again an induction. One can check the base case by hand. Then note that if a is the coefficient of $[x^i]L(x^n)$ then, calculating similarly to the previous claim we obtain that for each j, the coefficient of $x^j \otimes x^{i-j}$ in $\Delta(L(x^n))$ can be calculated directly and via the 1-cocycle property. This gives the following

$$\begin{aligned} a\binom{i}{j} &= \binom{n}{j}[x^{i-j}]L(x^{n-j}) \\ &= \binom{n}{j}\frac{(n-j)(n-j-1)\cdots(n-i+2)}{(i-j)!}[x]L(x^{n-i+1}) \\ &= \frac{n(n-1)\cdots(n-i+2)}{i!}[x]L(x^{n-i+1}) \end{aligned}$$

independently of j as desired.

Finally, the third claim is that with the coefficients as above, with free linear coefficient and 0 constant term, this gives all 1-cocycles.

The proof of this third claim is essentially the same calculations as above. We always checked all coefficients where a given coefficient appeared and they were consistent. This accounted for all coefficients, so the linear coefficient is free, and the constant term is incompatible with the 1-cocycle property.