COMBINATORIAL HOPF ALGEBRAS, WINTER 2020, ASSIGNMENT 1

SOLUTIONS

(1) (a) In undergrad abstract algebra you probably worked with ideals of rings (and maybe other things too, but lets stick with that for this problem). A K-algebra is a ring over K, so let's just translate the undergrad ideal of a ring definition, which says that I is a two sided ideal of A if $a, b \in I$ implies $a + b \in I$ and $r \in A$, $a \in I$ implies $ra \in I$ and $ar \in I$, where the multiplication is being denoted by concatenation. By the unit map, we see K inside A and so taking the special case of $r \in K \subseteq A$ along with the additivity shows that I is a subspace of A. $m(A \otimes I) = \{ra : r \in A, a \in I\} \subseteq I$ by the elementary definition. $m(I \otimes A) = \{ar : r \in A, a \in I\} \subseteq I$ by the elementary definition. Therefore we get the above definition of ideal.

In the other direction if we have an ideal as above, then for $r \in A$, $a \in I$, since $m(A \otimes I) \subseteq I$ we have $ra \in I$ and since $m(I \otimes A) \subseteq I$ we have $ar \in I$. Since I is a subspace we have additivity in I. Therefore we get the elemenetary definition of ideal.

(b) If $u \in \ker(f)$ and $v \in V$ then $(f \otimes f)(u \otimes v) = f(u) \otimes f(v) = 0 \otimes f(v) = 0$, and $(f \otimes f)(v \otimes u) = f(v) \otimes f(u) = f(v) \otimes 0 = 0$. Therefore $\ker(f) \otimes V + V \otimes \ker f \subseteq \ker f \otimes f$.

There is a usual trick for the other direction. Let $U = \ker(f) \otimes V + V \otimes \ker f$, we just checked that this is inside $\ker(f \otimes f)$ and it is clearly linear, so we can mod out by it. In particular, we have $f \otimes f : V \otimes V \to \mathrm{Im} f \to \mathrm{Im} f$ and hence this descends to $\overline{f \otimes f} : (V \otimes V)/U \to \mathrm{Im}(f) \otimes \mathrm{Im}(f)$.

If we can show that $\overline{f \otimes f}$ is an isomorphism then we're done since the first isomorphism theorem for vector spaces tells us $(V \otimes V)/\ker(f \otimes f) \cong \mathrm{Im}(f \otimes f)$ (this is actually just the rank-nullity theorem, since any two finite dimensional vector spaces of the same dimension are isomorphic, but this way is phrasing it in the style of an isomorphism theorem).

To show this, we'll exhibit the inverse map. Clearly the inverse map needs to be $g : \mathrm{Im}(f) \otimes \mathrm{Im}(f) \to (V \otimes V)/U$ defined by $g(f(v) \otimes f(w)) = v \otimes w + U$. The only question is whether this is actually well defined. So let's check. Suppose $f(v_1) \otimes f(w_1) = f(v_2) \otimes f(w_2)$ then $f(v_1) = f(v_2)$ and $f(w_1) = f(w_2)$. For g to agree defined on either of these arguments, we need $v_1 \otimes w_1 - v_2 \otimes w_2 \in U$. Calculate

$$\begin{aligned} v_1 \otimes w_1 - v_2 \otimes w_2 &= (v_1 - v_2) \otimes w_1 + v_2 \otimes (w_1 - w_2) \\ &\in \ker(f) \otimes A + A \otimes \ker(f) = U \end{aligned}$$

as desired.

(c) Coalgebra morphisms are first linear maps, so the kernel is a subspace.

Take $c \in C$ such that $f(c) = 0$. Then $(f \otimes f)(\Delta(c)) = \Delta(f(c)) = \Delta(0) = 0$. So $\Delta(c) \in \ker(f \otimes f)$ By the previous point we know $\ker(f \otimes f) = \ker(f) \otimes A + A \otimes \ker(f)$, so $\Delta(c) \in \ker(f) \otimes A + A \otimes \ker(f)$.
Additionally $\epsilon_D \circ f = \epsilon_C$ so $\epsilon_C(c) = \epsilon_D(f(c)) = \epsilon_D(0) = 0$.
Together this gives that $\ker(f)$ is a coideal.

(d) Consider $c \in C$ and $j \in J$. We need to show that any representative of c in C/J has the same coproduct and counit, so Δ and ϵ are well defined on C/J. After that, all the properties are automatic from C since extra identities can't make a diagram that already commutes fail to commute.
To show that any representatives of c in C/J have the same coproduct and counit it suffices to show c and $c + j$ have the same coproduct and counit. But the coproduct and counit are both linear, so $\Delta(c + j) - \Delta(c) = \Delta(j) \subseteq C \otimes J + J \otimes C = 0$ in C/J and $\epsilon(c+j) - \epsilon(c) = \epsilon(j) = 0$. This proves the result.

(e) First let's show that $\mathrm{Im}(f)$ inherits a coalgebra structure from D. The only thing to check for this point is that $\Delta_D(\mathrm{Im}(f)) \subseteq \mathrm{Im}(f) \otimes \mathrm{Im}(f)$. Take $c \in C$ and write $\Delta(c) = \sum_{(c)} c_1 \otimes c_2$. Then $\Delta_D(f(c)) = (f \otimes f)\Delta_C(c) = \sum_{(c)} f(c_1) \otimes f(c_2) \in \mathrm{Im}(f) \otimes \mathrm{Im}(f)$, as desired.
Then f itself gives a map $f : C/\ker(f) \to \mathrm{Im}(f)$. This map is well defined because it is linear and because of what a kernel is, i.e. if we have c and j as in the previous part then $f(c + j) - f(c) = f(j) = 0$.
$f : C \to D$ is a coalgebra morphism so $f : C/\ker(f) \to \mathrm{Im}(f)$ is also a coalgebra morphism by the same commutative diagrams as taking a quotient cannot break the commutativity of a diagram.
As a linear map $f : C/\ker(f) \to \mathrm{Im}(f)$ is one-to-one and onto hence is a vector space isomorphism and has an inverse as a linear map. Call the inverse g. The last thing to check is that g is also a coalgebra map, but this is also automatic as,

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"'] \\ C \otimes C \arrow[r, "f \otimes f"] & D \otimes D \end{tikzcd}$$

and g the vector space inverse implies

$$\begin{tikzcd} C \arrow[d, "\Delta_C"'] & D \arrow[l, "g"'] \arrow[d, "\Delta_D"'] \\ C \otimes C & D \otimes D \arrow[l, "g \otimes g"'] \end{tikzcd}$$

and likewise

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\epsilon_C"] & & D \arrow[dl, "\epsilon_D"] \\ & K & \end{tikzcd} \quad \text{implies} \quad \begin{tikzcd} C \arrow[dr, "\epsilon_C"] & & D \arrow[ll, "g"] \arrow[dl, "\epsilon_D"] \\ & K & \end{tikzcd}$$

This completes the proof.

(2) (a) Suppose $S^{2k-1} = \mathrm{id}$ for some $k \geq 1$. Then $S^{2k-1}(ab) = ab$. Now S is an antiautomorphism, so $S(ab) = S(b)S(a)$, $S(S(ab)) = S(S(b)S(a)) = S^2(a)S^2(b)$ and so on swapping a and b each time (prove it inductively if you like) to get $S^{2k-1}(ab) = S^{2k-1}(b)S^{2k-1}(a) = ba$ since the order is $2k-1$. Therefore $ab = ba$ and so H is commutative. But we showed that commutative Hopf algebras have antipodes of order at most 2. The only possiblity then is that S has order 1, that is $S = \mathrm{id}$.

If $S = \mathrm{id}$ then $\mathrm{id} \star \mathrm{id} = u\epsilon$. If $a \in \ker \epsilon$ then this says that $0 = \sum_{(a)} a_1 a_2$ so by the result from class the order is 1 or 2.

Note that order 1 can occur but only in very trivial situations, eg the field as a Hopf algebra over itself.

(b) (i) Call the ideal I. Calculate

$$
\begin{aligned}
\Delta(xy-1) &= \Delta(x)\Delta(y) - \Delta(1) \\
&= (x \otimes x)(y \otimes y) - 1 \otimes 1 \\
&= xy \otimes xy - 1 \otimes 1 \\
&= (xy-1) \otimes xy + 1 \otimes (xy-1) \\
&\in I \otimes A + A \otimes I
\end{aligned}
$$

Similarly

$$
\Delta(xy-1) = (y \otimes y)(x \otimes x) - 1 \otimes 1 = (yx-1) \otimes yx + 1 \otimes (yx-1) \in I \otimes A + A \otimes I
$$

Now take $a \in A$.

$$
\begin{aligned}
\Delta(a(xy-1)) &= \Delta(a)\Delta(xy-1) \\
&= \sum_{(a)} a_1(xy-1) \otimes a_2 xy + \sum_{(a)} a_1 \otimes a_2(xy-1) \\
&\in I \otimes A + A \otimes I
\end{aligned}
$$

since I is an ideal. Similar arguments hold for $a(yx-1)$, $(xy-1)a$, and $(yx-1)a$. Finally, Δ is linear so Δ applied to any linear combination of $xy-1$ and $yx-1$ lives in $I \otimes A + A \otimes I$.

Additionally, $\epsilon(xy-1) = 1-1 = 0$, $\epsilon(yx-1) = 1-1 = 0$, and similarly to the above, since ϵ is an algebra homomorphism, all the other elements of the ideal also map to 0.

Therefore I is a coideal.

(ii) First note that $S(xy-1) = S(y)S(x) - 1 = xy - 1 = 0$ and similarly $S(yx-1) = 0$ so S is well defined on the quotient.

Using $xy = 1$ and $yx = 1$,

$$\begin{aligned}
(S \star \mathrm{id})(x) &= S(x)x = yx = 1 = u\epsilon(x) \\
(\mathrm{id} \star S)(x) &= xS(x) = xy = 1 = u\epsilon(x) \\
(S \star \mathrm{id})(y) &= S(y)y = xy = 1 = u\epsilon(y) \\
(\mathrm{id} \star S)(y) &= yS(y) = yx = 1 = u\epsilon(y) \\
(S \star \mathrm{id})(z) &= S(1)z + S(z)x = z - zyx = 0 = u\epsilon(z) \\
(\mathrm{id} \star S)(z) &= S(z) + zS(x) = -zy + zy = 0 = u\epsilon(z)
\end{aligned}$$

so S behaves as an antipode on the generators. If a, b are words in x, y, z then $(S\star\mathrm{id})(ab) = \sum_{(a),(b)} S(b_1)S(a_1)a_2b_2$, so inductively if S behaves as an antipode on a and b, then summing the (a) sum gives a factor $u\epsilon(a)$ which pulls out and then summing the (b) sum gives $u\epsilon(b)$. Therefore inductively S behaves as an antipode on any words in the generators. This suffices to check that S is an antipode on account of linearity.
Now we need to determine the order of S. $S(x) = y$, $S(y) = x$ so $S(S(x)) = x$ and $S(S(y)) = y$. It remains to consider the order on z:

$$\begin{aligned}
S(z) &= -zy \\
S(S(z)) &= -S(y)S(z) = xzy \\
S(S(S(z))) &= S(y)S(z)S(x) = -xzy^2
\end{aligned}$$

Continuing, (formally by induction) one can prove $S^k(z) = (-1)^k x^{\lfloor k/2 \rfloor} z y^{\lceil k/2 \rceil}$. Since our only relations at this point are $xy = yz = 1$, these expressions are never equal to z and hence S has infinite order.

(iii) Calculate

$$\begin{aligned}
\Delta(x^n - 1) &= x^n \otimes x^n - 1 \otimes 1 \\
&= (x^n - 1) \otimes (x^n - 1) + 1 \otimes (x^n - 1) + (x^n - 1) \otimes 1 \\
&\subseteq \langle x^n - 1 \rangle \otimes H + H \otimes \langle x^n - 1 \rangle
\end{aligned}$$

and

$$\epsilon(x^n - 1) = \epsilon(x)^n - \epsilon(1) = 1 - 1 = 0$$

(iv) The thing we need to check to know that S passes to the quotient correctly is $S(\langle x^n - 1 \rangle) \subseteq \langle x^n - 1 \rangle$. So, compute

$$\begin{aligned}
S(x^n - 1) &= S(x)^n - S(1) \\
&= y^n - 1 \\
&= y^n - x^n y^n \\
&= (1 - x^n) y^n \subseteq \langle x^n - 1 \rangle
\end{aligned}$$

Now, finally we need to check the order of S. The previous calculations for $S(x)$ and $S(y)$ still hold as does the expression for $S(z)$. Consider

$$S^k(z) - z = (-1)^k x^{\lfloor k/2 \rfloor} z y^{\lceil k/2 \rceil} - z$$

For $k < 2n$ this expression is nonzero as no relation can remove the power of x. On the other hand for $k = 2n$

$$\begin{aligned} S^{2n}(z) - z &= x^n z y^n - z \\ &= z y^n - z \\ &= 0 \end{aligned}$$

where we used that $xy = 1$ and $x^n = 1$ implies $x^n = x^n y^n$ so $y^n = 1$. This proves the result.

(3) (a) This is in section 12 of Schmitt's "Incidence Hopf algebras" paper. Let $\mathcal{G}$ be a class of graphs closed under induced subgraph and disjoint union. Note also that at this point these are not graphs up to isomorphism, but just graphs. Isomorphism will come in later. All simple graphs will do the job and then we'll get exactly the first graph Hopf algebra from class. (If you're worried about this being too big take all simple graphs with vertices given by a subset of a fixed countable set.)

For $G \in \mathcal{G}$ consider the poset of subsets of $V(G)$ ordered by inclusion. Consider all products of intervals of such posets. This family of posets is interval closed and hereditary by definition.

The key is in the choice of Hopf relation. Define $[U_1, V_1] \times \cdots \times [U_n, V_n] \sim [W_1, X_1] \times \cdots \times [W_n, X_n]$ iff for all i, $G_i[V_i - U_i]$ and $H_i[X_i, W_i]$ are isomorphic as graphs, where G_i is the graph for which $[U_i, V_i]$ is an interval in its poset of vertex subsets and the same for H_i with respect to $[X_i, W_i]$.

This is an order compatible relation because intervals which are equivalent are necessarily isomorphic as posets. This is a congruence because direct product of posets corresponds (up to isomorphism) to disjoint union of graphs, and if we have two isomorphic graphs, then taking the disjoint union of each by another graph still results in isomorphic graphs. This relation is reduced because all induced graphs on 0 vertices are isomorphic. Therefore it is Hopf and we have an incidence Hopf algebra.

The product of the incidence Hopf algebra is poset product, which as discussed in the previous paragraph is disjoint union of graphs, as desired. We've observed many times that the unit and counit are the ones we keep seeing.

The coproduct on $[\emptyset, V(G)]$ for some $G \in \mathcal{G}$ is

$$[\emptyset, V(G)] = \sum_{S \subseteq V(G)} [\emptyset, S] \otimes [S, V(G)]$$

but the Hopf relation tells us intervals are the same if the difference in vertex sets induces the same graph, so we can consider the class of G to be $[\emptyset, V(G)]$ and we get

$$[G] = \sum_{S \subseteq V(G)} G[S] \otimes G[V(G) - S]$$

which is our first graph Hopf algebra.

(b) (i) The binomial Hopf algebra has a generator x in degree 1. This must correspond to the graph on one vertex. Then x^n is the graph on n vertices with no edges, and these are the only graphs we need to get the binomial Hopf algebra.

(ii) Let x_n be as in the question. As an algebra the Hopf algebra we are asking about will be the polynomial algebra $K[x_1, x_2, \ldots]$. Now calculate the coproduct.

$$\begin{aligned}\Delta(x_n) &= \sum_{S \subset V(K_n)} K_n[S] \otimes K_n[V(K_n) - S] \\ &= \sum_{S \subset V(K_n)} K_{|S|} \otimes K_{|V(K_n) - S|} \\ &= \sum_{i=0}^{n} \binom{n}{i} K_i \otimes K_{n-i}\end{aligned}$$