COMBINATORIAL HOPF ALGEBRAS, WINTER 2020, ASSIGNMENT 1

DUE THURSDAY FEB 13 IN CLASS

(1) Let A be a K-algebra and I a subspace of A. We say I is a (two sided) ideal if $m(A \otimes I) \subseteq I$ and $m(I \otimes A) \subseteq I$.

Let C be a K-coalgebra and J a subspace of C. We say that J is a (two-sided) coideal if $\Delta(J) \subseteq C \otimes J + J \otimes C$ and $\epsilon(J) = 0$.

  (a) Check that this definition of ideal lines up with the elementary definition you saw in undergrad abstract algebra.
  (b) Let $f : V \to W$ be a linear map. Prove that $\ker(f \otimes f) = \ker(f) \otimes V + V \otimes \ker(f)$.
  (c) Let $f : C \to D$ be a coalgebra morphism. Prove that $\ker(f)$ is a coideal.
  (d) Prove that if J is a coideal of a coalgebra C then C/J inherits a coalgebra structure.
  (e) Prove the first isomorphism theorem for coalgebras: if $f : C \to D$ is a coalgebra morphism then $\mathrm{Im}(f) \cong C/\ker(f)$ as coalgebras.

(2) Let H be a Hopf algebra and S the antipode of H. Write $S^k = \underbrace{S \circ S \circ \cdots \circ S}_{k \text{ times}}$. If

$$S^k = \mathrm{id}$$

then say S has order k. We showed that if H is commutative or cocommutative then S has order at most 2.

  (a) Prove that for any Hopf algebra S cannot have odd order > 2.
  (b) In fact all even orders (and infinite order) are possible. The next parts step you through a construction. Let A be $\mathrm{Span}_K(\{x, y, z\}^*)$ with the concatenation product, or equivalently, the free noncommutative algebra generated by x, y, z. Define a coproduct and counit on A by

$$\Delta(x) = x \otimes x, \qquad \Delta(y) = y \otimes y, \qquad \Delta(z) = 1 \otimes z + z \otimes x$$
$$\epsilon(x) = 1, \qquad \epsilon(y) = 1, \qquad \epsilon(z) = 0,$$

  and extending linearly and multiplicatively. A is a bialgebra. *You should check this, but I won't make you hand it in.*

    (i) Check that the ideal $\langle xy - 1, yx - 1\rangle$ is also a coideal. *This implies that $H = A/\langle xy - 1, yx - 1\rangle$ is a bialgebra (using the previous question).*
    (ii) Check that $S(x) = y$, $S(y) = x$ and $S(z) = -zy$ defines an antipode on H and prove that S has infinite order.
    (iii) Check that the ideal $\langle x^n - 1\rangle$ is also a coideal. *This implies that $J = H/\langle x^n - 1\rangle$ is a bialgebra.*
    (iv) Show S as defined above also gives an antipode for J and that it has order $2n$.

(3) (a) Show that the first graph Hopf algebra we defined is an incidence Hopf algebra.
  (b) The same construction works for any class of graphs which is closed under induced subgraphs and disjoint union.

(i) Give an example of such a class of graphs for which this graph Hopf algebra is isomorphic to the binomial Hopf algebra.

(ii) Consider the class $\mathcal{K}$ of graphs given by disjoint unions of complete graphs. Give a description of this Hopf algebra for $\mathcal{K}$ which does not involve any graphs. Use x_n as the name for the class of the complete graph K_n.