NCSym.

CO739, Winter 2020

Noncommutative symmetric functions another way

Say $f \in \boxed{K\langle\langle x_1, x_2, \ldots \rangle\rangle}$ of finite degree is a noncommutative symmetric function if permuting the variables leaves f unchanged.

Eg $\quad \sum_{i \neq j} x_i x_j x_i$

Let NCSym be the set of noncommutative symmetric functions in $K\langle\langle x_1, x_2, \ldots \rangle\rangle$.

Monomial noncommutative symmetric functions

Given a set partition of $\{1, 2, \ldots, n\}$,

Eg $\{1,3\}, \{2,4\}, \{5\}$ a set partition of $\{1,2,3,4,5\}$

Have monomial noncommutative symmetric functions

Eg

$$m_{\{1,3\},\{2,4\},\{5\}} = \sum_{1 \le i \ne j \ne k} x_i x_j x_i x_j x_k$$

first and 3rd position

2nd and 4th position

5th position

$$= x_1 x_2 x_1 x_2 x_3 + x_2 x_1 x_2 x_1 x_3 + x_3 x_1 x_3 x_1 x_2 + x_1 x_8 x_1 x_8 x_5 + x_{99} x_2 x_{99} x_2 x_{1000000} + \cdots$$

NCSym as a Hopf algebra

Then $\{m_\pi\}_{\pi \text{ set partition}}$ is a basis for NCSym.

NCSym is a Hopf algebra with series multiplication and $\Delta(f(\underline{x})) = f(\underline{y}, \underline{z})$ coproduct. As always