

First commutative square.

CO739, Winter 2020

Permutations return

Recall the Hopf algebra of permutations we discussed early on.

- Start with the set $\mathcal{P} = \bigcup_{n \geq 0} \mathcal{P}_n$ where \mathcal{P}_n is the set of permutations of $\{1, 2, \dots, n\}$.
- As a vector space the Malvenuto Reutenauer Hopf algebra of permutations is $H_{MR} = \text{Span}_K(\mathcal{P})$.
- Product shuffles with the second permutation given the higher numbers
- Coproduct is cut and normalize.

Take shuffle-deconcatenate
Hopf alg of permutations
in 1-line notation but
adjusting so permutation everywhere

Eg

$$21 \cdot 1 = 21 \cup 3 = \underline{\underline{321+231+213}}$$

$$\Delta(2314) = 2314 \otimes 1 + 231 \otimes 1 + \overset{1}{\downarrow} 2 \otimes 12 + 1 \otimes 213 + 1 \otimes 2314$$

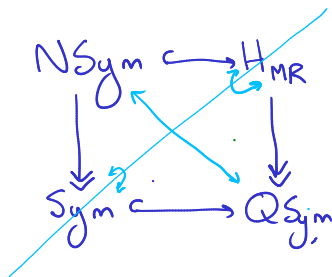
A square relating some Hopf algebras we've seen

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Example continued

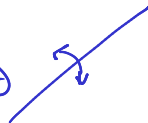
Then we have a square

It turns out



And duality is

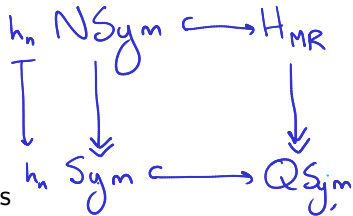
reflection along



Descent sets

Let's write down the missing maps.

Given $\sigma \in \mathcal{P}_n$ the descent set of σ is



$$\text{Des}(\sigma) = \{1 \leq \boxed{i} \leq n-1 : \underline{\sigma_i} > \underline{\sigma_{i+1}}\}$$

Eg $\text{Des}(56231874) = \{2, 4, 6, 7\}$

Think of these as cut points and build a composition (cut just after the index)



and the composition is the sizes of the chunks

so $\text{comp}(\text{Des}(56231874)) = 2, 2, 2, 1, 1$

Composition of the descent set

Treat the descent set as cut points in $1\ 2\ 3\ 4\ \dots\ n$

Eg *eg* $1\ 2\ | 3\ 4\ | 5\ 6\ | 7\ | 8$
 $\cdot\ \cdot\ | \cdot\ \cdot\ | \cdot\ \cdot\ | \cdot\ | \cdot$

The sizes of the parts give a composition. Call this the composition of the descent set, $\text{comp}(\text{Des}(\sigma))$.

$$\text{so } \text{comp}(\text{Des}(56231874)) = 2, 2, 2, 1, 1$$

The surjection

Malvenuto showed

$$H_{MR} \rightarrow \text{QSym}$$
$$\sigma \mapsto F_{\text{comp}(\text{Des}(\sigma))}$$

is a surjective Hopf morphism, where

$$F_\alpha = \sum_{\beta \geq \alpha} M_\beta$$

(fundamental quasisymmetric functions), and \geq is refinement/coarsening

the big side is the coarser side

so eg $(2, 1, 5, 1) \geq (1, 1, 1, 3, 2, 1)$

Another self-duality

Also, H_{MR} is self-dual via

$$H_{MR}^{\circ} \rightarrow H_{MR}$$

$$\sigma^* \mapsto \sigma^{-1}$$

and so from the previous slide we get the dual map $\text{NSym} \rightarrow H_{MR}$,
 and all these together give the square.

$$\begin{array}{ccc}
 \text{NSym} & \xleftarrow{f^*} & H_{MR} \\
 \downarrow h_n & & \downarrow \circlearrowleft \\
 \text{Sym} & \xleftarrow{\text{incl. no formal power series}} & \text{QSym} \\
 \downarrow h_n & & \downarrow \text{F}_{\text{comp}}(\text{Des}(\circlearrowleft))
 \end{array}$$