First commutative square.

CO739, Winter 2020

Permutations return

Recall the Hopf algebra of permutations we discussed early on.

- Start with the set $\mathcal{P} = \bigcup_{n \geqslant 0} \mathcal{P}_n$ where $\mathcal{P}_n$ is the set of permutations of $\{1, 2, \ldots, n\}$.
- As a vector space the Malvenuto Reutenauer Hopf algebra of permutations is $\boxed{H_{MR}} = \mathrm{Span}_K(\mathcal{P})$.
- Product shuffles with the second permutation given the higher numbers
- Coproduct is cut and normalize.

Take shuffle - deconcatenate Hopf alg of permutations in 1-line notation but adjusting so permutations everywhere

Eg

$21 \cdot 1 = 321 + 231 + 213$

$= 21 ⧢ 3$

$\Delta(2314) = 2314 \otimes 1\!\!1 + 231 \otimes 1 + 12 \otimes 12 + 1 \otimes 213 + 1\!\!1 \otimes 2314$

Example continued

Then we have a square

It turns out

$$\begin{tikzcd} NSym \arrow[r, hook] \arrow[d, two heads] & H_{MR} \arrow[d, two heads] \\ Sym \arrow[r, hook] & QSym \end{tikzcd}$$

And duality is reflection along

Descent sets

$$\begin{tikzcd} h_n \ \mathrm{NSym} \arrow[r, hook] \arrow[d, mapsto] \arrow[d, two heads, shift left=4] & H_{MR} \arrow[d, two heads] \\ h_n \ \mathrm{Sym} \arrow[r, hook] & \mathrm{QSym}, \end{tikzcd}$$

Let's write down the missing maps.

Given $\sigma \in \mathcal{P}_n$ the *descent set* of σ is

$$\mathrm{Des}(\sigma) = \{1 \leq \boxed{i} \leq n-1 : \sigma_i > \sigma_{i+1}\}$$

Eg $\mathrm{Des}(56231874) = \{2, 4, 6, 7\}$

1 2 3 4 5 6 7 8

Think of these as cut points and build a composition (cut just after the index)

eg 1 2 | 3 4 | 5 6 | 7 | 8

and the composition is the sizes of the chunks

so $\mathrm{comp}(\mathrm{Des}(56231874)) = 2, 2, 2, 1, 1$

Composition of the descent set

Treat the descent set as cut points in $1\ 2\ 3\ 4\ \cdots\ n$

Eg   eg $1\ 2 \mid 3\ 4 \mid 5\ 6 \mid 7 \mid 8$

The sizes of the parts give a composition. Call this the composition of the descent set, comp(Des(σ)).

so $\operatorname{comp}(\operatorname{Des}(56231874)) = 2, 2, 2, 1, 1$

The surjection

Malvenuto showed

$$H_{MR} \to \mathrm{QSym}$$
$$\sigma \mapsto F_{\mathrm{comp}(\mathrm{Des}(\sigma))}$$

is a surjective Hopf morphism, where

$$F_\alpha = \sum_{\beta \geq \alpha} M_\beta$$

(fundamental quasisymmetric functions), and $\geq$ is refinement/coarsening

the big side is the coarse side

so eg $(2,1,5,1) \geq (1,1,1,3,2,1)$

Another self-duality

Also, H_{MR} is self-dual via

$$\begin{aligned} H_{MR}^{*} &\to H_{MR} \\ \sigma^* &\mapsto \sigma^{-1} \end{aligned}$$

and so from the previous slide we get the dual map $\mathrm{NSym} \to H_{MR}$, and all these together give the square.

$$\begin{tikzcd} h_n \arrow[d, mapsto] & \mathrm{NSym} \arrow[r, hook, "f^*"] \arrow[d, two heads] & H_{MR} \arrow[d] & \sigma \arrow[d, mapsto, "f"] \\ h_n & \mathrm{Sym} \arrow[r, hook, "\text{incl into formal power series}"'] & \mathrm{QSym} & F_{\mathrm{comp}(\mathrm{Des}(\sigma))} \end{tikzcd}$$