Duality between QSym and NSym.

CO739, Winter 2020

Duality

Proposition

QSym and NSym are graded dual Hopf algebras

proof: Use the pairing

$$QSym \times NSym \longrightarrow K$$

$$\langle M_\alpha, h_\beta \rangle = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

Now just need to verify

$$\langle \Delta(M_\alpha), h_\beta \otimes h_\gamma \rangle = \langle M_\alpha, h_\beta h_\gamma \rangle$$

So let's compute to show this

$$\langle \Delta(M_\alpha), h_\beta \otimes h_\gamma \rangle$$

same as for sym. fns $x_1 x_2 \cdots \mapsto y_1 y_2 \cdots z_1 z_2 \cdots$

$$= \langle \sum_{\alpha = \delta\varepsilon} M_\delta \otimes M_\varepsilon, h_\beta \otimes h_\gamma \rangle$$

concatenate

$$= \begin{cases} 1 & \alpha = \beta\gamma \\ 0 & \text{otherwise} \end{cases}$$

$$\langle M_\alpha, h_\beta h_\gamma \rangle$$

$$= \langle M_\alpha, h_{\beta\gamma} \rangle$$

concatenate of compositions

$$= \begin{cases} 1 & \alpha = \beta\gamma \\ 0 & \text{otherwise} \end{cases}$$

Observations

This duality is compatible with the self-duality of Sym.

Specifically, the map

$$\mathrm{Sym} \hookrightarrow \mathrm{QSym}$$

is dual to the map

$$K\langle h_1, h_2, \ldots\rangle \qquad K[h_1, h_2 \ldots]$$

$$\begin{array}{ccc} \mathrm{NSym} & \twoheadrightarrow & \mathrm{Sym} \\ h_n & \mapsto & h_n \end{array}$$

but now the h_n's commute

~~where~~

Because the pairing is the same

Note Sym is both commutative and cocommutative

QSym is commutative but is not cocommutative ← coprod is given by deconcatenation of indexing composition

NSym is not commutative but is cocommutative ← manifestly by def.