Categorification of combinatorial Hopf algebras.

CO739, Winter 2020

We don't need no stinkin' underlying combinatorial objects

You've categorified a structure if you've obtained it from some categorical construction.
Specifically, as representations of something.

The combinatorial objects are no longer inputs to your construction but arise from the construction.

If you've done it right you get new insight into the concrete objects as well as having a nice new abstract structure.

Towers

If you like thinking of symmetric functions as representations of the symmetric group then you already like categorification.
The key is that you have a tower of groups $\{S_n\}$.

As well as Sym, the Malvenuto Reutenauer Hopf algebra of permutations can also be categorified by representations of a tower of groups. and others ....

To step beyond Sym you can move from towers of groups to towers of algebras. (QSym and NSym can be done this way.)
Another way to step beyond Sym is be more general in what you allow the representation theory to be. With supercharacter theory you can build NCSym. and more.

powerful mainstream approach

Other ways

Towers aren't the only way.
Matt Szczesny spoke in the algebraic combinatorics seminar a month or so ago on an approach via Hall algebras.
The Connes-Kreimer Hopf algebra and it's friends can also be categorified operadically.

Don't ask me, ask these people

Some of our York colleagues are experts on this. Check out this talk http://garsia.math.yorku.ca/NantelTalk/Lothar2017_Bergeron.pdf
Here's an introduction from a thesis
https://amypang.github.io/notes/cha_thesis4pt1.pdf

References for the specific facts I mentioned

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