The chromatic symmetric function.

an example of the previous theorem giving something interesting.

CO739, Winter 2020

Return to the first Hopf algebra of graphs

Let H be the first Hopf algebra of graphs.
Multiplication is disjoint union.

$$\Delta(G) = \sum_{W \subseteq V(G)} G[W] \otimes G[V - W].$$

Let

$$\zeta(G) = \begin{cases} 1 & \text{if } G \text{ is only isolated vertices} \\ 0 & \text{otherwise.} \end{cases}$$

What is ψ?

What is $\psi(G)$?

$$\psi(G) = \sum_{\lambda} \zeta_{\lambda}(G) m_{\lambda}$$

But what is $\zeta_{\lambda}(G)$?

What is ζ_λ?

let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$

$$\zeta_\lambda(G) = (\zeta_{\lambda_1} * \zeta_{\lambda_2} * \cdots * \zeta_{\lambda_k})(G)$$

$$= \sum_{\substack{W_1 \cup \cdots \cup W_k = V(G) \\ \text{the } W_i \text{ disj.} \\ W_i \text{ has } \lambda_i \text{ vertices}}} \zeta_{\lambda_1}(G[W_1])\, \zeta_{\lambda_2}(G[W_2]) \cdots \zeta_{\lambda_k}(G[W_k])$$

= # of partitions of $V(G)$ into $W_1, W_2, \ldots, W_k$ with W_i having λ_i vertices with no edges in $G[W_1], G[W_2], \ldots, G[W_k]$

= # partitions of $V(G)$ into colour classes of sizes $\lambda_1, \lambda_2 \ldots$ with no edge having both ends the same colour

= # proper colourings of G using colour 1 exactly λ_1 times, colour 2 exactly λ_2 times etc.

Chromatic symmetric function

So

$$\psi(G) = \sum_{\substack{\text{proper colourings} \\ \kappa: V(G) \to \mathbb{Z}_{\geq 1}}} \prod_{v \in V(G)} x_{\kappa(v)}$$

which is the *chromatic symmetric function*.

You'll probably hear more about chromatic symmetric functions starting in the fall with Sophie and Logan's arrival.