Third definition of combinatorial Hopf algebras.

CO739, Winter 2020

Characters

Given a Hopf algebra H over K, a *character* of H is an algebra morphism $\zeta : H \to K$.

A particularly important character for QSym is

$$\begin{aligned} \zeta_Q : \mathrm{QSym} &\to K \\ f(x_1, x_2, x_3, \ldots) &\mapsto f(1, 0, 0, \ldots) \end{aligned}$$

Compare with Feynman rules

Feynman rules should also be algebra maps

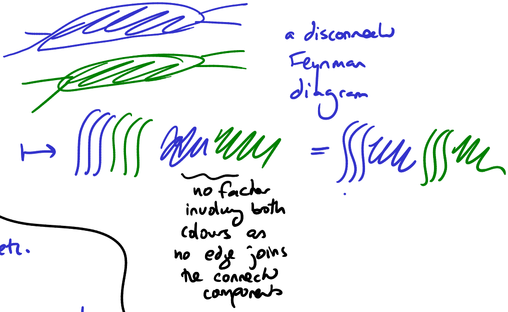
multiplicative

Target space not K.

But from an actual experiment you'd get a number (i.e. an elt of K)

because you evaluate external momenta, masses etc. at fixed values

So if you fix values of parameters it ultimately becomes a character.



Furthermore you want your Feynman rules to be Hopf characters

Comes down to a nice convolution property. Say one external parameter L (think $\log(s)=L$ in toy model)

$F: \mathcal{H} \to K[L]$, want $F|_{L=L_1} * F|_{L=L_2} = F|_{L=L_1+L_2}$

Third definition of Combinatorial Hopf algebra

The third definition of Combinatorial Hopf algebra is:

A Combinatorial Hopf algebra is a pair (H, ζ) of a graded connected Hopf algebra H over K with each H_n finite dimensional and a character $\zeta : H \to K$.

Examples

We hadn't thought about characters before, but we can typically pick a fairly trivial ζ and get something good.

Eg first graph Hopf algebra (mult disjoint union, $\Delta(G) = \sum_{W \subseteq V(G)} G[W] \otimes G[V-W]$)

A nice character is

$$\zeta(G) = \begin{cases} 1 & \text{if } G \text{ is just some isolated vertices} \\ 0 & \text{otherwise.} \end{cases}$$

What are the morphisms for this definition

What should a morphism of combinatorial Hopf algebras $\psi : (H_1, \zeta_1) \to (H_2, \zeta_2)$ mean?

Answer: $\psi : H_1 \to H_2$ is a Hopf alg morphism

and
$$\begin{tikzcd} H_1 \arrow[rr, "\psi"] \arrow[dr, "\zeta_1"'] & & H_2 \arrow[dl, "\zeta_2"] \\ & K & \end{tikzcd}$$
commutes.

Universal property

Theorem (Aguiar, Bergeron, Sottile)

$(QSym, \zeta_Q)$ is the terminal object in the category of combinatorial Hopf algebras (in the sense of definition 3).
That is, for any combinatorial Hopf algebra (H, ζ) there is a unique morphism ψ of combinatorial Hopf algebras,
$\psi : (H, \zeta) \to (QSym, \zeta_Q)$.

proof Write ζ_n for ζ restricted to H_n

then $\zeta_n \in H_n^*$ and hence $\zeta_n \in H^\circ$

We know $QSym^\circ = NSym = K\langle h_1, h_2 \dots \rangle$

So there is a unique alg map

$$\phi : NSym \longrightarrow H^\circ$$
$$h_n \longmapsto \zeta_n$$

since NSym is free

Proof

continued

If is also a Hopf alg. map

we know $\Delta(h_n) = \sum_{i=0}^{n} h_i \otimes h_{n-i}$. What about $\Delta(\zeta_n)$?

ζ is a character so it is multiplicative

so as a map its coproduct will go by divided powers, lining up with h_n.

Thus we have the dual map

$\psi : H \to QSym$

Claim ψ is the map we want

check this. • ψ is a Hopf alg map.

(from ϕ an alg map get ψ a coalg map and ζ character gives the multiplicativity)

• Now for $g \in H$

$$\langle \phi(h_n), g \rangle = \langle h_n, \psi(g) \rangle$$

$$\overset{\shortparallel}{\langle \zeta_n, g \rangle}$$

Proof continued

so $h_n(\Psi(g)) = \zeta_n(g) = \begin{cases} \zeta(g) & g \in H_n \\ 0 & \text{otherwise} \end{cases}$

h_n sends M_n to 1

sends M_λ to 0 for $\lambda \neq n$

But this is ζ_Q restricted to degree n.

So in degree n $\zeta_Q(\Psi(g)) = \zeta_n(g) = \zeta(g)$

and this holds for all n so $\zeta_Q(\Psi(g)) = \zeta(g)$.

So Ψ has the appropriate properties to be the map we want.

Formula for ψ

The formula for ψ is, for $g \in H$

$$\psi(g) = \sum_{c \text{ composit}} \langle h_c, \psi(g) \rangle M_c$$

$$= \sum_{c \text{ composit}} \zeta_c(g) M_c \quad \text{where } \zeta_c = \zeta_{c_1} * \zeta_{c_2} * \dots * \zeta_{c_k}$$

$$c = (c_1, c_2, c_3, \dots c_k)$$

there are ζ restricted to c_k-graded pieces

same as if sum is over c composites of $|g|$ because if not must be 0.

Theorem

For any cocommutative combinatorial Hopf algebra (H, ζ) there is a unique morphism ψ of combinatorial Hopf algebras, $\psi : (H, \zeta) \to (Sym, \zeta_S)$, where ζ_S is evaluation at $(x_1, x_2, \ldots) = (1, 0, 0, \ldots)$.

The proof is the same since $\mathrm{Sym}^* = K[h_1, h_2, \ldots]$ is free commutative.

The formula for ψ is, by the same argument,

$$\psi(g) = \sum_\lambda \zeta_\lambda(g) m_\lambda$$

as before could sum over λ partitions of $|g|$
since other terms are all 0.