

# A commutative hexagon.

CO739, Winter 2020

# The Grossman Larson Hopf algebra

The graded dual of Connes-Kreimer is also a named Hopf algebra.

The Grossman Larson Hopf algebra of rooted trees is a Hopf algebra defined on  $\text{Span}_K(\mathcal{T})$ , trees not forests.

Product is grafting the subtrees of the left tree onto the right tree.

Eg

$$\begin{aligned}
 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \cdot \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} &= \begin{array}{c} \color{green} \diagup \quad \diagdown \\ \color{green} \bullet \quad \bullet \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ \color{green} \bullet \quad \bullet \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \color{green} \bullet \end{array} \\
 &= \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + 2 \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}
 \end{aligned}$$

$$\begin{aligned}
 \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \cdot \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= \begin{array}{c} \color{green} \bullet \\ \color{green} \diagup \quad \color{grey} \diagdown \\ \color{green} \bullet \quad \color{grey} \bullet \end{array} + \begin{array}{c} \color{grey} \bullet \\ \color{grey} \diagup \quad \color{green} \diagdown \\ \color{grey} \bullet \quad \color{green} \bullet \end{array} + \begin{array}{c} \color{green} \bullet \\ \color{green} \diagup \quad \color{grey} \diagdown \\ \color{green} \bullet \quad \color{grey} \bullet \end{array} + \begin{array}{c} \color{grey} \bullet \\ \color{grey} \diagup \quad \color{grey} \diagdown \\ \color{grey} \bullet \quad \color{grey} \bullet \end{array} \\
 &= \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + 2 \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}
 \end{aligned}$$

# Grossman Larson coproduct

The unit for the multiplication is  $\bullet$  (not empty tree which is in  $\mathcal{T}$ )

Coproduct is deshuffle subtrees, giving each side a root.

$$\begin{aligned}
 \text{Eg } \Delta(\text{tree}) &= \bullet \otimes \text{tree} + \text{tree} \otimes \bullet \\
 &+ \text{tree}_1 \otimes \text{tree}_2 + \text{tree}_2 \otimes \text{tree}_1 + \text{tree}_3 \otimes \bullet \\
 &+ \bullet \otimes \text{tree}_4 + \text{tree}_4 \otimes \bullet + \text{tree}_5 \otimes \text{tree}_6
 \end{aligned}$$

It is graded and connected and so a Hopf algebra in the usual way.

↳ by number of edges

degree 0 is  $\text{Span}_K(\bullet) \cong K$

# Duality

Grossman-Larson is dual to Connes-Kreimer, though exactly how is a little subtle.

*ie there is an isomorphism*

First, remove the root in Grossman-Larson, then both are defined on trees, and the operations of Grossman-Larson still make sense.

$$\begin{array}{l}
 \begin{array}{ccc}
 \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} & \rightsquigarrow & \bullet \bullet \\
 \Delta(\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}) = \begin{array}{l} \bullet \otimes \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \otimes \bullet \\ + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \otimes \bullet + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \otimes \bullet \end{array} & \rightsquigarrow & \begin{array}{l} 1 \otimes \bullet + \bullet \otimes 1 \\ + \bullet \otimes \bullet + \bullet \times 1 \end{array} = \Delta(\bullet \bullet) \\
 \bullet \bullet = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} & \rightsquigarrow & \bullet \bullet + \bullet = \bullet \bullet
 \end{array} \\
 \left. \begin{array}{l} \text{Connes-Kreimer} \\ \bullet \bullet \\ \bullet \bullet = \bullet \bullet \quad \bullet \bullet \bullet \\ \bullet \bullet \bullet = \bullet \bullet \bullet \\ \Delta(\bullet \bullet) = \begin{array}{l} 1 \otimes 1 + 1 \otimes \bullet \\ + \bullet \otimes \bullet \\ \Delta(\bullet \bullet) = \bullet \bullet \otimes 1 + 1 \otimes \bullet \bullet \\ 2 \bullet \bullet \bullet \end{array} \end{array} \right\}
 \end{array}$$

Then the pairing is not the obvious one; there are some multiplicities.

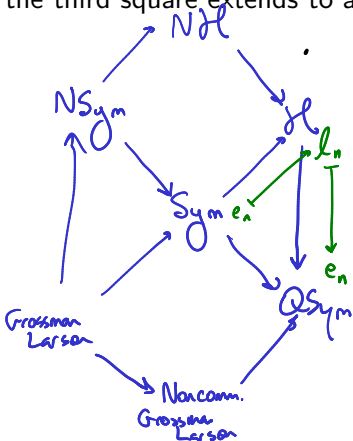
*Not correct in first published proof corrected by Mike Hoffman.*

# Noncommutative Grossman-Larson

The same story works noncommutatively giving a noncommutative version of Grossman-Larson and the graded dual to  $N\mathcal{H}$ .

# Extending the square

Then the third square extends to a hexagon



← what is this map?

We know a way to  
 make maps  $\mathcal{H} \rightarrow \mathbb{Z}^n$

→ use  $B_+$  and the  
 . universal property of  $(\mathcal{H}, B_+)$

So let's do it

# Extra space

So what is the 1-cycle on  $\mathcal{QSym}$

$$A_+ : \mathcal{QSym} \rightarrow \mathcal{QSym}$$

$$c \rightarrow M_{c,1} \quad \text{concatenate}$$

Need to check  $A_+$  is a 1-cycle

$$\begin{aligned} \Delta A_+(M_c) &= ((id \otimes A_+) \Delta + A_+ \otimes 1)(M_c) \\ &= \sum_{d+c=1} M_d \otimes M_c \\ &= \sum_{a+b=c} M_a \otimes M_{b,1} + M_{c,1} \otimes 1 = (id \otimes A_+) \Delta(M_c) + A_+(M_c) \otimes 1. \end{aligned}$$

So automatically get  $Z^* : \mathcal{H} \rightarrow \mathcal{QSym}$

$$Z^* B_+(f) = A_+(Z^*(f))$$

Finally what does  $Z^*$  do to  $Q_i = \underbrace{B_+(B_+(B_+(B_+ \dots (1))))}_{i \text{ } B_+'s} = A_+(A_+ \dots (1)) = M_{(1, \dots, 1)}$