A commutative hexagon.

CO739, Winter 2020

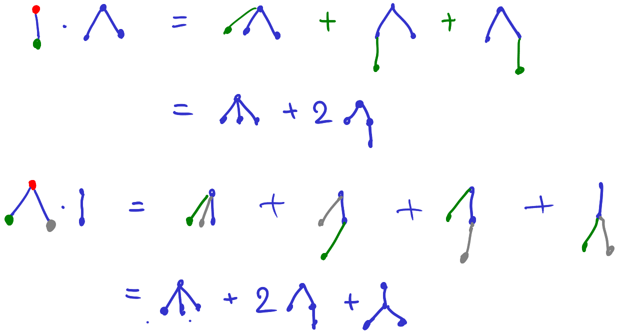
The Grossman Larson Hopf algebra

The graded dual of Connes-Kreimer is also a named Hopf algebra.

The *Grossman Larson Hopf algebra* of rooted trees is a Hopf algebra defined on $\mathrm{Span}_K(\mathcal{T})$, trees not forests.

Product is grafting the subtrees of the left tree onto the right tree.

Eg

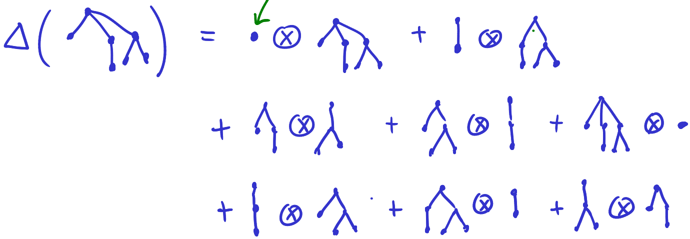


Grossman Larson coproduct

The unit for the multiplication is $\bullet$ (not empty tree which isn't in $\mathcal{T}$)

Coproduct is deshuffle subtrees, giving each side a root.

Eg



It is graded and connected and so a Hopf algebra in the usual way.

↳ by number of edges

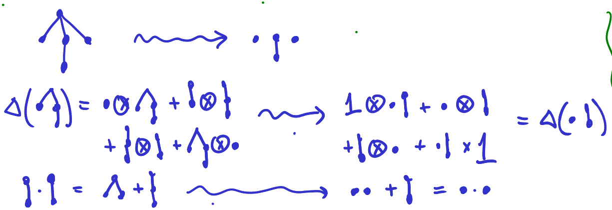
degree 0 is $\mathrm{Span}_K(\bullet) \cong K$

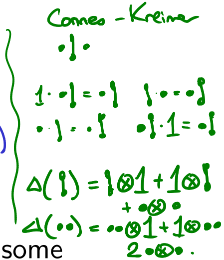
Duality

Grossman-Larson is dual to Connes-Kreimer, though exactly how is a little subtle.

ie there is an isomorphism

First, remove the root in Grossman-Larson, then both are defined on trees, and the operations of Grossman-Larson still make sense.





Then the pairing is not the obvious one; there are some multiplicities.

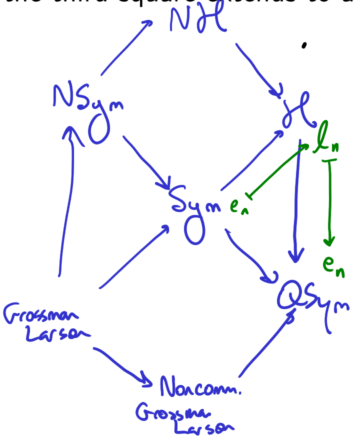
Not correct in first published proof corrected by Mike Hoffman.

Noncommutative Grossman-Larson

The same story works noncommutatively giving a noncommutative version of Grossman-Larson and the graded dual to $N\mathcal{H}$.

Extending the square

Then the third square extends to a hexagon



← what is this map?

We know a way to make maps $\mathcal{H} \longrightarrow$?

→ use B_+ and the universal property of $(\mathcal{H}, B_+)$

So lets do it

Extra space

So what is the 1-cocycle on QSym

$$A_+ : QSym \to QSym$$
$$c \mapsto M_{c\cdot 1}$$

concatenate

Need to check A_+ is a 1 cocycle

$$\Delta A_+(M_c) = \sum_{d\cdot e = c\cdot 1} M_d \otimes M_e$$

$$((id \otimes A_+)\Delta + A_+ \otimes 1)(M_c)$$

$$\|$$

$$= \sum_{a\cdot b = c} M_a \otimes M_{b\cdot 1} + M_{c\cdot 1} \otimes 1 = (id \otimes A_+)\Delta(M_c) + A_+(M_c) \otimes 1.$$

So automatically get $Z^* : \mathcal{H} \to QSym$

$$Z^* B_+(f) = A_+(Z^*(f))$$

Finally what does Z^* do to $\ell_i = \underbrace{B_+(B_+(B_+(B_+ \ldots (1))}_{i\ B_+s} = A_+(A_+ \ldots (1)) = M_{(1,1,1,1,1,1)}$.