MIDTERM INSTRUCTIONS AND QUESTIONS TO PREPARE

CO 430/630 WINTER 2026

Instructions

For the midterm you will solve two questions from the following list on the board with Jerónimo and me as the audience. You will select one and I will select one. For each of them you will be asked follow ups that go beyond the question itself but that should be doable based on having solved the question.

The exam will take place in MC 6483. You should sign up for a timeslot at `https://calendar.app.google/F4D3aqB3XwR6nAqS6`

If you have any questions about the format don't hesitate to ask me.

Questions

(1) We had $B(x,y) = \sum_{n\geq 0} \binom{y}{n} x^n \in \mathbb{Q}[y][[x]]$. Prove that $B(x,y)B(x,z) = B(x,y+z)$. Be sure to prove any preliminary formal power series facts that you need which we didn't prove in class or on assignment 1.

(2) Prove that $\prod_{k\geq 0}(1 + x^{2^k}) = \sum_{n\geq 0} x^n$ as formal power series.

(3)
  (a) Suppose $T(x)$ is a formal power series satisfying $T(x) = x\exp(T(x))$ and let c be any invertible element of the ring we are working over. Show that

$$[x^n]\exp(cT(x)) = \frac{c(c+n)^{n-1}}{n!}$$

  You may use usual exp rules for the formal power series.

  (b) Describe a class of rooted trees (labelled or unlabelled) that has as counting sequence the sequence given in the previous part for $c = 2$.

(4) Explain via a decomposition why (unlabelled) plane rooted trees counted by vertices, (unlabelled) full binary trees (vertices have 0 or 2 children) counted by leaves, and noncrossing rooted chord diagrams counted by one more than the number of chords are all equinumerous.

(5) Consider ternary strings with no 012, 021, 102, 120, 201, 210 consecutive substrings. What is the ordinary generating series for such strings counted by length.

(6) A *derangement* is a permutation with no fixed points.
  (a) Give a species-theoretic relationship between the species of derangements, the species $\mathcal{E}$ and the species $\mathcal{S}$.
  (b) Give a species-theoretic relationship between the species of derangements, the species $\mathcal{E}$ and the species $\mathcal{C}$.

(7) Consider the species $\mathcal{O} = \mathcal{C}[\mathcal{L}_{\geq 1}]$
  (a) Describe this species in elementary terms and draw an illustrative example.
  (b) Show that $O(x) = C(2x) - C(x)$ in two different ways.
  (c) Show there are $(n-1)!(2^n - 1)$ of these for any X with $|X| = n$.