CO 430/630 LECTURE 8 SUMMARY PART 2

WINTER 2026

Summary

We continued lecture 8 with a naive take on labelled objects. We started with an example, rooted trees with no plane structure. We drew the first few on the board and then drew them again but now *labelling* them in all possible ways, that is each vertex gets a number from 1 to the number of vertices and no two get the same number. Of the two rooted trees on three vertices, the one that is a line has 6 labellings while the one that is a root with two children has only three. You thought a little about bigger examples and noticed that the max number of labellings of a rooted tree on n vertices is $n!$ and the minimum number is n. We also observed that we can think of combinatorial objects as labelled objects in sometimes less obvious ways. For example a permutation can be seen as a labelling of dots on a line.

The right notion of generating series for labelled objects is the *exponential generating series* so for a labelled combinatorial class $\mathcal{A}$ the exponential generating series is

$$A(x) = \sum_{a \in \mathcal{A}} \frac{x^{w(a)}}{w(a)!} = \sum_{n \geq 0} \frac{a_n}{n!} x^n$$

I use the same notation convention as for ordinary generating series so you use the context to tell them apart (and I should be careful to make that context clear).

We did two examples, permutations $\mathcal{S}$ with exponential generating series $S(x) = \frac{1}{1-x}$ and (nonempty) cycles $\mathcal{C}$ with exponential generating series $C(x) = \log(1/(1-x))$ The fact that $S(x) = \exp(C(x))$ is not a coincidence and we'll return to it a bit later.

That was the first pass of labelled counting. It's a bit unsatisfying though, what exactly are we labelling? We can improve matters in our second pass by working with combinatorial classes *built of atoms* (as we'd discussed when working with the pointing operator). Given an unlabelled combinatorial class $\mathcal{A}$ that's built of atoms (and using it's atom counting weight), a *labelling* of $a \in \mathcal{A}$ is a bijection from the atoms of a to $\{1, \dots, w(a)\}$. This gives us a more precise notion of labelling and we'll run with this for another hour or so before we decide "built of atoms" is still too much of a kluge and move to combinatorial species.

Next time

Next time we'll pursue labelled operations from the built of atoms perspective and then begin combinatorial species.

References

Chapter II of Flajolet and Sedgewick's book *Analytic Combinatorics* is a good place to look for a conventional (not species-theoretic) description of labelled counting, though not exactly in the form I discussed it.