CO 430/630 LECTURE 8 SUMMARY PART 1

WINTER 2026

Summary

We started by finishing up with the transfer matrix method. From last class we'd had $D_{ij}(x)$ the generating series of walks from i to j counted by length and had an expression in terms of matrix inverse. Then using Cramer's rule we have

$$\begin{aligned} D_{ij}(x) &= \left((1 - Ax)^{-1}\right)_{ij} \\ &= (-1)^{i+j}\frac{\det(1 - Ax; i, j)}{\det(1 - Ax)} \end{aligned}$$

where the $;j,i$ thing is notation for removing the ith row and jth column, we won't need that notation much so it's ok that it's a little clunky.

A few notes. We see the characteristic polynomial in the denominator, we also see that $D_{ij}(x)$ is a rational function in x with degree at most one less than the degree of the char poly in the numerator.

Closed walks are extra convenient in this context because they are the diagonal entries, so we can obtain the generating series of closed walks (with any start vertex, but not allowing empty walks), call it $C(x)$, as a trace $C(x) = \sum_{n\geq 1} \operatorname{tr}(A^n)x^n$, then using the fact that if $\lambda_1, \ldots, \lambda_d$ are the nonzero eigenvalues then $\operatorname{tr}(A^n) = \lambda_1^n + \cdots + \lambda_d^n$ then

$$C(x) = \frac{\lambda_1 x}{1 - \lambda_1 x} + \cdots + \frac{\lambda_k x}{1 - \lambda_k x} = \frac{-x\frac{d}{dx}\det(1 - Ax)}{\det(1 - Ax)}$$

where the last equality is from putting things on a common denominator.

This technique is good for any time where you build the next step out of the previou step (or a fixed number of previous steps) in possibly multiple ways. As an example we did ternary strings with no 11 or 23. We noticed that the closed walks count such strings where the condition is forbidden in a cyclic way (watch out, you need to add the bit when you cross the edge, not when you're on the vertex for that to make sense.)

The lecture continued with the start of labelled counting which is in a separate summary file.

References

Stanley's Enumerative Combinatorics Chapter 4.7.