CO 430/630 LECTURE 7 SUMMARY

WINTER 2026

Summary

Well, we had a snow day on Monday, so this is only lecture 7 even though it was the Wednesday.

We recalled the multiset operator from last time and then recognized that by chaning some signs we can get the set operator instead: Let $\mathcal{B} = \mathsf{Set}(\mathcal{A})$ be the combinatorial class of sets of elements of $\mathcal{A}$ then

$$\mathsf{Set}(\mathcal{A}) \leftrightharpoons \prod_{a\in\mathcal{A}} \{\epsilon, a\}$$

so

$$B(x) = \prod_{a\in\mathcal{A}} 1 + x^{w(a)} = \prod_{n\geq 1} (1+x^n)^{a_n}$$

and as before we can continue to rewrite

$$\begin{aligned}
B(x) &= \prod_{n\geq 1}(1+x^n)^{a_n} \\
&= \exp\left(\log\left(\prod_{n\geq 1}(1+x^n)^{a_n}\right)\right) \\
&= \exp\left(\sum_{n\geq 1} a_n \log(1+x^n)\right) \\
&= \exp\left(\sum_{n\geq 1} a_n \sum_{i\geq 1}(-1)^{i-1}\frac{x^{ni}}{i}\right) \\
&= \exp\left(\sum_{i\geq 1} \frac{(-1)^{i-1}}{i}\left(\sum_{n\geq 1} a_n x^{ni}\right)\right) \\
&= \exp\left(\sum_{i\geq 1}(-1)^{i-1}\frac{A(x^i)}{i}\right)
\end{aligned}$$

The sign can be a bit of a pain, but we wrote out the first few terms of this and the multiset case to see how it works out; there's cancellation in this case, and a sort of augmentation in the multiset case.

If we wanted the generating series for cycles of elements from the class $\mathcal{C}$ then we'd have

$$\sum_{i\geq 1}\frac{\phi(i)}{i}\log\left(\frac{1}{1-C(x^i)}\right)$$

where ϕ is the Euler phi function. We didn't prove this because it will be on assignment 2 (probably).

We also discussed pointing. For this we want to talk about combinatorial classes that are *built of atoms* that is which have a decomposition in terms of operations we've seen along with recursive appearances of itself and explicit elelments of weight 0 and weight 1 (the last of those are the *atoms*). We can also do likewise with decompositions that are systems of equations and that still counts as built of atoms. If you find this a little unsatisfying, wait two weeks for the species.

We'll be interested in the weight function that comes from counting the atoms.

Most of the examples we've worked with are built of atoms, eg rooted trees counted by number of vertices or binary string counted by length.

Now if $\mathcal{C}$ is built of atoms then we can define a new combinatorial class $\mathcal{C}^\bullet$ whose elements are pairs of an element of $\mathcal{C}$ and an atom of that element. Usually we draw these just by drawing the element of $\mathcal{C}$ and using an arrow or a circle to mark the chosen atom.

Then

$$C^\bullet(x) = \frac{xd}{dx}C(x)$$

which we can prove by direct calculation

$$C^\bullet(x) = \sum_{\substack{(c,z)\\ c\in\mathcal{C}\\ z \text{ atom of } c}} x^{w(c)} = \sum_{c\in\mathcal{C}} w(c)x^{w(c)} = \frac{xd}{dx}C(x)$$

We also mentioned that composition came up on the LIFT sheet and more genrally corresponds to putting a structure of one class on another, which will come back later.

Next we did a few examples.

For the last third of the class, we started in on the transfer matrix method. The transfer matrix method starts with a little piece of algebraic graph theory. Suppose D is a diagraph on v vertices and assign a weight to each arc (or just let the weights be 1 when you don't want weights). The *adjacency matrix* of D, which we'll denote A is the matrix who's i, jth entry is $w(e)$ if e is the arc from i to j and 0 if no such arc exists. (If you want to allow multiple arcs between two vertices, then sum the weights.)

The key proposition from the start of algebraic graph theory is that the i, jth entry of A^n is the sum of the weights of all directed walks of m arcs from i to j where the weight of a walk is the product of the weights of the arcs making it up. In particular if the weights are all 1 then the i, jth entry of A^n is the *number* of directed walks of m arcs from i to j. The proof is basically matrix multiplication; write it out.

This is nice for enumeration because if we let $D_{ij}(x)$ be the generating series for walks from i to j counted by length then

$$D_{ij}(x) = \sum_{n\geq 0}(A^n)_{ij}x^n = \left(\sum_{n\geq 0} A^n x^n\right)_{ij} = \left((1-Ax)^{-1}\right)_{ij}$$

and this inverse makes sense.

Next time

Next time we'll finish our brief exploration of the transfer matrix method and hopefully start labelled counting.

References

Chapter I of Flajolet and Sedgewick's book *Analytic Combinatorics* gives a good presentation of the material on operations on combinatorial classes in much this language.

A good reference for the transfer matrix method is Stanley's Enumerative Combinatorics Chapter 4.7.