CO 430/630 LECTURE 3 SUMMARY

WINTER 2026

Summary

We started by using Hensel's lemma to prove the condition for when a formal power series has a compositional inverse.

Proposition 1. *$A(x) \in R[[x]]_+$ has a compositional inverse iff $[x]A(x)$ is invertible.*

Proof. (sketch) The real work is a good example of how Hensel's lemma is used. For the $\Leftarrow$ direction let $F(t,x) = t - A(x)$, check the hypotheses of Hensel's lemma are satisfied and so obtain $B(t) \in R[[t]]_+$ such that $0 = F(t, B(t)) = t - A(B(t))$. This gives one side of the inverse. The other side is always true in this context. One nice way to show it is to define $\mathrm{ev}_C : R[[x]] \to R[[x]]$ by $\mathrm{ev}_C(A(x)) = A(C(x))$ and show that this map is injective for $C \neq 0$ (use val!). For the $\Rightarrow$ direction consider the coefficient of x in $B(A(x)) = x$. We gave all the details in class. □

Next we talked about some other kinds of formal series. The two we discussed are formal Laurent series and formal Dirichlet series, though I briefly mentioned formal Puisseaux series and formal series in noncommuting variables.

Definition 2. *The* ring of formal Laurent series $R((x))$ *over* R *is*

$$R((x)) = \{\sum_{n \geq N} a_n x^n : N \in \mathbb{Z}, a_n \in R\}$$

with $\sum_{n \geq N_1} a_n x^n + \sum_{n \geq N_2} b_n x^n = \sum_{n \geq \min\{N_1, N_2\}} (a_n + b_n) x^n$ *and* $\left(\sum_{n \geq N_1} a_n x^n\right)\left(\sum_{n \geq N_2} b_n x^n\right) = \sum_{n \geq N_1 + N_2} \left(\sum_{i \in \mathbb{Z}} a_i b_{n-i}\right) x^n$ *where we take the convention that* $a_i = 0$ *for* $i < N_1$ *and* $b_i = 0$ *for* $i < N_2$*, which also guarantees that the inner sum in the definition of product is a finite sum.*

Much of what we discussed still works, coefficent extraction (but useful to define $[x^{-1}]A(x)$ as the formal residue operator), val (but it can be negative). If F is a field then $F((x))$ is also a field.

There are two cautions, you don't want two way infinite series because you don't want a calculation like $\sum_{n \in \mathbb{Z}} x^n = \sum_{n \geq 0} x^n + \sum_{n \geq 0} x^{-n-1} = \frac{1}{1-x} + \frac{x^{-1}}{1-x^{-1}} = 0$. Secondly, the difference between $R((x))((y))$ and $R((y))((x))$ is important (and it wasn't for formal power series. To see this is we expand $(x+y)^{-1}$ in $\mathbb{Q}((x))((y))$ we get $\sum_{n \geq 0} (-1)^n y^n x^{-n-1}$ but get it with x and y flipped which is quite a different expression if we expand in $\mathbb{Q}((y))((x))$.

This is a good time to state the Lagrange implicit function theorem (LIFT)

Theorem 3. *Suppose $\mathbb{Q} \subseteq R$ and $\phi(\lambda) \in R[[\lambda]]$ is invertible. Then*

- *there exists a unique $A(x) \in R[[x]]_+$ such that $A(x) = x\phi(A(x))$*
- *$[x^n]A(x) = \frac{1}{n}[\lambda^{n-1}]\phi(\lambda)^n$ for $n \geq 1$.*

- *for any* $f(\lambda) \in R((\lambda))$

$$[x^n]f(A(x)) = \frac{1}{n}[\lambda^{n-1}]f'(\lambda)\phi(\lambda)^n \qquad \text{for } n \neq 0$$
$$[x^=]f(A(x)) = [\lambda^0]f(\lambda) + [\lambda^{-1}]f'(\lambda)\log(\phi(\lambda)/\phi(0))$$

where the second term of the last expression is 0 *when* $val_\lambda(f(\lambda)) \geq 0$.

Note that the second point is the special case of the third point when $f(\lambda) = \lambda$. It's nice to state separately since the second point is the one you'll use most often in combinatorial applications (but not always!). See the CO330 notes (linked in the references below) for a proof using the formal residue operator. We'll do a combinatorial proof in class next week.

Definition 4. *The* ring of formal Dirichlet series *over* R *is the set of expressions of the form* $\sum_{n\geq 1} a_n n^{-s}$ *with* $\sum_{n\geq 1} a_n n^{-s} + \sum_{n\geq 1} b_n n^{-s} = \sum_{n\geq 1}(a_n+b_n)n^{-s}$ *and* $\left(\sum_{n\geq 1} a_n n^{-s}\right)\left(\sum_{n\geq 1} b_n n^{-s}\right) = \sum_{n\geq 1}\left(\sum_{k|n} a_k b_{n/k}\right) n^{-s}$

Dirichlet series can be useful for counting things where the weight of a product of two objects is the product of the weights rather than the sum of the weights as you'd expect for ordinary generating series (which are coming right up!). The prototypical example is the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

which is the Dirichlet generating series for the positive integers.

We finished the class by setting up ordinary generating series.

Definition 5.
- *A* combinatorial class $\mathcal{A}$ *is a finite or countably infinite set with a weight function* $w : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ *such that* $\mathcal{A}_n = \{a \in \mathcal{A} : w(a) = n\}$ *is finite for all* n.
- *The* counting sequence *of* $\mathcal{A}$ *is* $a_0, a_1, a_2, \ldots$ *where* $a_i = |\mathcal{A}_i|$.
- *The ordinary generating series of* $\mathcal{A}$ *is* $A(x) = \sum_{a\in\mathcal{A}} x^{w(a)} = \sum_{n\geq 0} a_n x^n \in \mathbb{Z}[[x]]$.

This is notation mostly in the style of Flajolet and Sedgewick's book (see references below), but you can use a different style if you prefer. Unless otherwise specified the letters will line up as above (curly for the class, capital roman for the generating series and lower case for the sequence) and I will use this convention without comment in the future.

We can also have multivariate ordinary generatin gseries with multiple weight functions $w_1 : \mathcal{A} \to \mathbb{Z}_{\geq 0}$, $w_2 : \mathcal{A} \to \mathbb{Z}_{\geq 0}$, $\ldots$, so $A(x_1, x_2, \ldots) = \sum_{a\in A} x_1^{w_1(a)} x_2^{w_2(a)} \cdots$. You still want each $\mathcal{A}_{n_1,n_2,\ldots}$ to be finite and you want finitely many w_i, or at least you want each a to only have finitely many nonzero w_i.

The point of this set up is to enumerate things by the symbolic method. There are four basic steps

- Pin down what $\mathcal{A}$ and w are.
- Find a way to decompose $\mathcal{A}$ so each element is decomposed into smaller or simpler pieces. The decomposition should give bijections that are weight preserving and behave appropriately on the standard operations (which are to come)
- Convert the decomposition into equations for the generating series
- Solve or manipulate or extract coefficient or otherwise analyse in order to ansewr the question at hand.

Next time

Next time we'll look at the operations, first the ones that come up in math 239 and then some more and also give examples.

References

Already mentioned are the videos from one of Kevin's offerings `https://www.math.uwaterloo.ca/~kpurbhoo/winter2021-co630/co630-videos.html`.

The CO 330 notes by David Wagner `https://uwaterloo.ca/combinatorics-and-optimization/sites/default/files/uploads/documents/co330-notes_0.pdf`

Chapter I of Flajolet and Sedgewick's book *Analytic Combinatorics* gives a good presentation of this combinatorial set up in much this language with lots of examples and in a way that's pretty practical for someone who wants to use it.