CO 430/630 LECTURE 1 SUMMARY PART 2

WINTER 2026

Summary

The first lecture continued with the start of content.

For us R will be an integral domain, that is a nonzero commutative ring with no zero divisors.

Definition 1. *$R[[x]]$ is the set of expressions of the form $\sum_{i\geq 0} a_i x^i$ for $a_i \in R$ with the operations*

$$\sum_{i\geq 0} a_i x^i + \sum_{i\geq 0} b_i x^i = \sum_{i\geq 0} (a_i + b_i) x^i$$

$$\left(\sum_{i\geq 0} a_i x^i\right)\left(\sum_{i\geq 0} b_i x^i\right) = \sum_{i\geq 0}\left(\sum_{j=0}^{i} a_j b_{i-j}\right) x^i$$

and where we view $R \subseteq R[[x]]$ via $r \mapsto rx^0 + 0x^1 + 0x^2 + \cdots$. $R[[x]]$ is the ring of formal power series *over R.*

The point is these are useful for enumeration and formally they are just symbols indexed by one way infinite series. There's another way to do the formalism as an inverse limit or completion of polynomials, which we briefly said a word about at the end of the lecture but which we won't go into in this course.

Some notation

Definition 2. *Let $A(x) = \sum_{i\geq 0} a_i x^i$ then we have coefficient extraction notation: $[x^n]A(x) = a_n$ and we define a valuation*

$$val_x(A(x)) = \begin{cases} \min\{m : [x^m]A(x) \neq 0\} & A(x) \neq 0 \\ \infty & A(x) = 0 \end{cases}$$

The point of val is to make rigorous what counts as valid in formal power series. It has good properties:

Proposition 3. *Let $A(x), B(x) \in R[[x]]$.*

(1) $val_x(A(x)) = \infty$ *iff* $A(x) = 0$
(2) $val_x(A(x) + B(x)) \geq \min\{val_x(A(x)), val_x(B(x))\}$
(3) $val_x(A(x)B(x)) = val_x(A(x)) + val_x(B(x))$

We checked these; it's straightforward. The min is because there could be cancellation of the lowest nonzero term. The valuation makes it really easy to prove

Proposition 4. *R an integral domain implies that $R[[x]]$ is an integral domain.*

To show no zero divisors just suppose there were one and apply val to each side.

As in any integral domain we can define

Definition 5. $\frac{A(x)}{B(x)} = C(x)$ *means* $A(x) = B(x)C(x)$.

Next time

Next time we will use val to define a topology on $R[[x]]$ and then continue with more on formal power series.

References

This is pretty standard, so you can find it in many places. Wikipedia has pretty good coverage (and will tell you more about the x-adic completion of polynomials approach too, if you're curious). Closer to this course here are notes and videos from one of Kevin's offerings `https://www.math.uwaterloo.ca/~kpurbhoo/winter2021-co630/co630-videos.html` which cover it in much the way I did.