CO 430/630 LECTURE 17 SUMMARY

WINTER 2026

Summary

We defined the convolution product and convolution algebra of linear maps $f : C \to A$ where C is a coalgebra and A is an algebra. This let us finally define a Hopf algebra, you need to have an antipode S which is the convolution inverse of id.

Then I gave some examples including polynomials with $\Delta(x) = x \otimes 1 + 1 \otimes x$, the Connes-Kreimer Hopf algebra of rooted trees, and group rings. This material is covered in more detail in lectures 2, 3, and 4 of the notes from the Combinatorial Hopf algebra topics course I taught some years ago, see links below. I very briefly and eclectically said something about the Hopf algebra of symmetric functions.

The next point is that if your bialgebra is graded (so it is a graded vector space and all the maps are graded) and connected (so the 0th graded piece is just the field) then the antipode is automatic and has a recursive formula $S(x) = -x - \sum_i S(x_{i,1})x_{i,2}$ where $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_i x_{i,1} \otimes x_{i,2}$.

An important example for this class is the incidence coalgebra. Let P be a locally finite poset and $\mathrm{Int}(P)$ the set of intervals of P. The *incidence coalgebra of P* is a coalgebra on $\mathrm{Span}_K(\mathrm{Int}(P))$ with

$$\Delta([x, y]) = \sum_{x \leq z \leq y} [x, z] \otimes [z, y]$$

and $\epsilon([x, y])$ is 1 if $x = y$ and 0 otherwise. The *reduced incidence coalgebra of P* is the same thing but working with intervals up to poset isomorphism.

You can do this more generally for suitable families of intervals and suitable equivalence relations on them. There is a classic paper of Schmitt on this, which is also ref'd below. If you take the dual (in the sense of functions to the base field) then you get the *incidence algebra*, which you might have seen if you were looking at the section in Stanley on Möbius inversion.

We make the *incidence Hopf algebra* from the incidence algebra with Cartesian product as the product and in the reduced case we get the antipode

$$S([x, y]) = \sum_{k \geq 0} \sum_{x = x_0 < x_1 < \cdots < x_k = y} (-1)^k \prod_{i=1}^{k} [x_{i-1}, x_i]$$

This relates to Möbius inversion by defining $\zeta([x, y]) = 1$ for all $[x, y] \in \mathrm{Int}(P)$ and extending linearly to $\mathrm{Span}_K(\mathrm{Int}(P))$. Then define μ to be the convolution inverse of ζ. Then we get

(1) $\mu = \zeta \circ S$
(2) $f = g * \zeta \Leftrightarrow g = f * \mu$ for f, g maps from the incidence coalgebra to K.

The proofs are calculations

(1) $\zeta * (\zeta \circ S) = (\zeta \circ \mathrm{id}) * (\zeta \circ S) = \zeta \circ (\mathrm{id} * S) = \zeta u \epsilon$ and evaluating $\zeta u\epsilon$ on $[x,x]$ and $[x,y]$ with $x \neq y$ we see that $\zeta u \epsilon = u\epsilon$ so $\zeta \circ S$ is the convolution inverse of ζ, hence is μ.

(2) $f * \mu = (g * \zeta) * \mu = g * (\zeta * \mu) = g * u\epsilon = g$

The point is that a nice formula for S gives a nice formula for μ and vice versa. In fact S essentially is μ.

That all tied things up very neatly except that I realized you want to know a little about q-counting for one of the worksheets.

The idea of q-counting is to replace a nonnegative integer by a polynomial in q where evaluating at $q=1$ gives the integer back. You want to do this in a nice way so that when the integer used to count something, now the polynomial counts the same something with an extra parameter or statistic.

The set up is $[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$, $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ and the q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$.

q-binomial cofficients have many binomial coefficient properties, eg

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

q-binomial coefficients count many things in just the way you'd hope. My favorite is that they count lattice walks from $(0,0)$ to (ℓ, k) by the area under the walk.

If you've heard talks about major index on permutations that's also this world because we have $\sum_{\sigma\in S_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} = [n]_q!$ where $\mathrm{inv}(\sigma)$ is the number of inversions in σ, that is, (i,j) such that $i<j$ and $\sigma(i) > \sigma(j)$ and $\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{Des}(\sigma)} i$ where $\mathrm{Des}(\sigma) = \{i : \sigma(i) > \sigma(i+1)\}$. This is just the beginning of a giant story that we aren't going to discuss in this course.

Next time

Next time you'll work on some worksheets.

References

Lectures 2, 3, and 4 of my past topics course on combinatorial Hopf algebras, see `https://www.math.uwaterloo.ca/~kayeats/teaching/co739_w20`. You might also look at lecture 5 for the grading and lectures 7 and 8 for the incidence stuff.

Incidence Hopf algebras, by William Schmitt.