

# CO 430/630 LECTURE 16 SUMMARY

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## SUMMARY

Here are three classic examples of Möbius inversion.

**Example 1.** Let  $P = B_n$ , then

$$\mu(S, T) = \begin{cases} (-1)^{|S|-|T|} & S \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

which you can see by a quick induction, or just from the binomial theorem.

This is inclusion-exclusion. How does that work exactly? Let  $S$  be a set of properties that elements of  $\mathcal{C}$  can have or not have. Let  $f(U)$  be the number of elements of  $\mathcal{C}$  satisfying exactly the properties of  $U \subseteq S$  and no others. This is often hard to count directly because of overlapping conditions. Let  $g(U)$  be the number of elements of  $\mathcal{C}$  satisfying at least the properties in  $U$  and potentially others as well. This is usually easier to compute. Then  $g(U) = \sum_{U \subseteq T} f(T)$  and so by Möbius inversion  $f(U) = \sum_{U \subseteq T} (-1)^{|T|-|U|} g(T)$ . Most often we want  $f(\emptyset) = \sum_{T \subseteq S} (-1)^{|T|} g(T)$ , which is your classic inclusion exclusion. Since you can't say this without doing derangements, we quickly counted derangements again in this form.

**Example 2.** If we take  $P$  to be positive integers ordered by divisibility then we get the classic number theoretic Möbius inversion. This is where the name Möbius inversion came from. By unique factorization you can think of a positive integer as the multiset of its prime factors, or equivalently as a multiplicity vector of the multiplicities of the prime factors.

It turns out that the Möbius function here is

$$\mu(i, j) = \begin{cases} (-1)^m & \text{if } j/i \text{ has distinct prime factors and } m \text{ is the number of them} \\ 0 & \text{otherwise} \end{cases}$$

since this depends only on  $j/i$  write  $\mu(j/i)$  for  $\mu(i, j)$ . Thinking in terms of the vectors of multiplicities this can be written as

$$\mu((a_1, a_2, \dots), (b_1, b_2, \dots)) = \begin{cases} (-1)^{\sum_i (b_i - a_i)} & \text{if each } b_i - a_i \text{ is 0 or 1} \\ 0 & \text{otherwise} \end{cases}$$

You can prove the original formula for  $\mu$  inductively, or the next example will make it even easier from the second formulation.

The classic number theoretic Möbius inversion then says  $g(n) = \sum_{d|n} f(d)$  for all  $n$  is equivalent to  $f(n) = \sum_{d|n} g(d) \mu(n/d)$  for all  $n$ .

**Example 3.** Let  $P$  be  $\mathbb{Z}_{\geq 0}$  with the usual total order. Then inductively

$$\mu(i, j) = \begin{cases} 1 & i = j \\ -1 & i + 1 = j \\ 0 & \text{otherwise} \end{cases}$$

The previous example is the product of copies of this example since the multiplicities behave this way, and that's the more structural way to understand the Möbius function for the previous example.

Anyway, back to this situation, Möbius inversion says  $g(n) = \sum_{i=0}^n f(i)$  for  $n \geq 0$  if and only if  $f(n) = g(n) - g(n-1)$  for  $n \geq 0$  (with the convention  $g(-1) = 0$ ). This is telescoping sums, or you can also see it as a discrete analogue of the fundamental theorem of calculus.

That finished our section strictly on Möbius inversion. Then we started our brief discussion of Hopf algebras (which will tie back to Möbius inversion next time). I taught a topics course on this some years ago, and we basically did a quick overview version of most of lectures 2 and 3, see links in the references section below. In particular, we defined concatenation, shuffle, deconcatenation, deshuffle on words, we defined algebra, coalgebra, and bialgebra and discussed how those relate to the word examples.

#### NEXT TIME

Next time we'll actually define Hopf algebras do some more examples of combinatorial Hopf algebras including incidence algebras which will tie us back to Möbius inversion.

#### REFERENCES

Stanley's Enumerative Combinatorics chapter 3.

Lectures 2 and 3 of my past topics course on combinatorial Hopf algebras.

- [https://www.math.uwaterloo.ca/~kayeats/teaching/co739\\_w20/lec2\\_summary.pdf](https://www.math.uwaterloo.ca/~kayeats/teaching/co739_w20/lec2_summary.pdf)
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